ON $^3$He-$^4$He DILUTION REFRIGERATORS

PART II

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Abstract

An analysis is presented of the superfluid injection dilution refrigeration process with emphasis on the case in which both $^4$He and $^3$He are circulated. Though the treatment is almost equivalent to that of Pennings et al., our analysis leads for $T \leq 0.03$ K to an explicit expression for the temperature distribution along the tube ending in the mixing chamber. The cooling power of the refrigerator can easily be determined. A representative numerical example is treated in some detail.

1. Introduction

In the foregoing paper 1) (which will be referred to here as I) a treatment was presented of $^3$He-$^4$He dilution refrigerators in which $^3$Re is circulated. The analysis resulted in a direct determination of cooling powers, lowest attainable temperatures and temperature distributions along tubes. In the present paper we treat the method of dilution refrigeration with superfluid injection 2-11), placing special emphasis on the method in which both $^3$Re and $^4$He are circulated. The main objective is to calculate (i) the temperatures in the neighbourhood of the chamber where the dilution process takes place and (ii) the cooling power of the devices.

The two superfluid injection methods to be discussed are again based on the observation that at temperatures below 0.87 K a phase separation occurs in a liquid $^3$He-$^4$He mixture into a $^3$He-rich and a $^3$He-poor phase, and that there is a difference in entropy per $^3$He atom in the two phases. Phase equilibrium at temperatures below 0.03 K implies a $^3$He-rich phase that is 100% pure and a $^3$He-poor phase with molar $^3$He concentration $X$ equal to $0.064 \pm 0.0005$.

In figure 1 a schematic drawing is given of a superfluid injection dilution refrigerator as designed by Taconis et al. 3). It shows two interconnected chambers M and $M_1$, a superleak $S_1$ through which $^4$He is brought into $M$, and a superleak $S_2$ through which $^4$He is withdrawn from $M_1$ (by definition a superleak passes only the superfluid component of a superfluid). Chamber $M$, the interconnecting tube $P$ and the upper part of $M_1$ are assumed to be filled with concentrated $^3$He (i.e. the $^3$He-rich phase), while the lower part of $M_1$ is filled with a dilute mixture ($^3$He-poor phase). An external pump delivers superfluid $^4$He to $M$ through $S_1$. Part of the concentrated $^3$He present in $M$ will be dis-
Fig. 1. Schematic drawing of a superfluid injection dilution refrigerator. Superfluid $^4$He is injected into chamber M via superleak $S_1$. Droplets of $^4$He enriched with $^3$He fall through tube P. The downward flow of $^3$He causes an upward flow of pure $^3$He of equal magnitude. There is a phase separation surface in chamber $M_1$ (demixing chamber). The $^4$He leaves $M_1$ via superleak $S_2$. The $^4$He circulation is for instance driven by a thermomechanical $^4$He pump. A $^3$He cryostat system ensures sufficient cooling of chamber $M_1$. Temperatures in $M_1$ are kept at $\leq 0.5$ K. Temperatures in M as low as 8.3 mK have been reached.

Solved into the injected $^4$He, as required by thermodynamic equilibrium considerations. This gives the desired cooling effect. The "enriched" $^4$He will then be transported from M to $M_1$. Several models can be suggested for this transport, and in fact more experiments should be performed to test the respective models. The model adopted here is that, for not too high $^4$He flow rate $\dot{n}_4$ the transport of $^4$He takes place in the form of isolated $^4$He droplets, all having a $^3$He concentration such that there is phase equilibrium with the sur-
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rounding $^3$He. These droplets fall down from $M$ to $M_1$ due to the gravitational force acting on the mass-density difference between the enriched $^4$He droplets and the surrounding fluid; no extra external pressure is assumed to force this $^4$He transport. The fall of droplets implies in turn a heat load on chamber $M_1$, as dilute $^3$He enters $M_1$, carrying considerably more entropy than the equal amount of concentrated $^3$He going upwards. For that reason it is necessary to cool chamber $M_1$ by means of a $^3$He cryostat. In practice, the temperature of $M_1$ will be about 0.5 K.

The double circulation dilution refrigeration method (fig. 2) is based on the same principle, but the necessary cooling system on $M_1$ is now replaced by an internal $^3$He circulating dilution refrigeration system, i.e. a $^3$He supply tube $P_1$ as well as a $^3$He exhaust tube $P_2$ have been added to $M_1$. In this way temperatures in $M_1$ can be kept well below 0.1 K leading to less severe restrictions on the cooling performance in chamber $M$. An additional advantage is that differences in the chemical potential $\mu_4$ between chambers $M$ and $M_1$ will be reduced, facilitating the application of a $^4$He thermomechanical pump (fountain pump) used to circulate the superfluid $^4$He. The $^4$He is withdrawn from the evaporation chamber $D$. Via $S_1$, $M$, $P$, $M_1$ and $P_2$ it is circulated through the system (see fig. 2).

In section 2 an energy balance equation is established for tube $P$ (see figs 1 and 2). The equation is a second-order differential equation for the function $T(l)$, where $T$ is the temperature and $l$ the tube length parameter. It is shown that an explicit analytical expression for $T(l)$ can be given in the low-temperature limit ($T \lesssim 0.03$ K). In section 3 energy balance equations applying to chambers $M$ and $M_1$ are established, serving as additional conditions imposed on the differential equation for the function $T(l)$. As we restrict the treatment to temperatures at or below 0.03 K, the additional condition imposed on $M_1$ can only be used in the double circulation case. A procedure is outlined to calculate the cooling power at the mixing chamber $M$. Results are presented for the function $T(l)$ as well as the cooling power. In section 4 some minimum temperature considerations are dealt with.

2. An energy balance equation for tube $P$

In deriving the temperature distribution in tube $P$ (see figs 1 and 2) in the steady state we have to consider in some detail the downward flow of the isolated droplets containing dilute superfluid $^3$He-$^4$He mixture and the compensating upward flow of the concentrated normal $^3$He-$^4$He mixture. We will restrict the treatment to temperatures at or below 0.03 K. In the case of fig. 1 this implies a restriction to the tube part not too far away from $M$. It enables us to use a number of simplifications in the temperature dependence of the
Fig. 2. Schematic drawing of a double circulation dilution refrigerator. There is a phase separation surface in chamber $M_1$. The $^3\text{He}$ is circulated via the supply tube $P_1$, the chamber $M_1$, the exhaust tube $P_2$ and the evaporation chamber $D$. The heat exchange system $H$ ensures sufficient precooling of the incoming $^3\text{He}$. The superfluid $^4\text{He}$ is injected into chamber $M$; the mechanism is the same as that indicated in fig. 1. Liquid $^4\text{He}$ circulates via the tube $P$ the chamber $M_1$, the tube $P_2$ and the evaporation chamber $D$. It is exhausted from $D$ by means of a thermomechanical $^4\text{He}$ pump via a superleak. Temperatures in $M_1$ can be kept as low as 0.03 K. Temperatures in $M$ have been reached \(^{11}\) as low as 4 mK.

thermal conductivities $\kappa_d(T)$ and $\kappa_c(T)$ of the dilute and concentrated mixtures, respectively, and in the specific heat at constant volume $C_{3d}(T)$ and $C_{3c}(T)$ of the two mixtures involved (see table I, where these dependencies are given, and a number of constants related to the above quantities are defined). Furthermore, we will put the molar concentration $X(T)$ of $^3\text{He}$ in the dilute mixture equal to $X = 0.064$. 

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TABLE I

Some low-temperature relations and parameter values

$$k_d = \kappa_d(T) \ T = 3 \times 10^{-4} \text{ J s}^{-1} \text{ m}^{-1} \quad T \lesssim 0.01 \text{ K}$$

$$k_c = \kappa_c(T) \ T = 3.3 \times 10^{-4} \text{ J s}^{-1} \text{ m}^{-1} \quad T \lesssim 0.03 \text{ K}$$

$$k \approx k_c$$

$$C_{3d}(T) \approx s_d(T) \approx a_2 \ T/m_3$$

$$C_{3e}(T) \approx s_{3e}(T) \approx a_3 \ T/m_3$$

$$a_2 = 108 \text{ J K}^{-2} \text{ mol}^{-1}$$

$$a_3 = 24 \text{ J K}^{-2} \text{ mol}^{-1}$$

$$X(T) = X(0)(1 + \beta T^2) \quad T \lesssim 0.15 \text{ K}$$

$$\beta = 10.8 \text{ K}^{-2}$$

$$X(0) = X = 0.064 \pm 0.0005$$

$$m_3 = 0.003 \text{ kg mol}^{-1}$$

$$m_4 = 0.004 \text{ kg mol}^{-1}$$

$$\rho_{3e} = 81.5 \text{ kg m}^{-3}$$

$$\nu_4 = 6.93 \times 10^{-3} \text{ kg}^{-1} \text{ m}^{3}$$

Let the cross-section of the tube be circular; let $l$ denote the length coordinate ($l = L_0$ in M; $l = L_1$ in $M_1$, $L_1 > L_0$) and let the diameter $d$ be independent of $l$. If $n_4$ is the flow rate of $^4$He (i.e. the number of moles $^4$He per second passing through a given cross-section) the accompanying $^3$He flow rate will be $n_{31} = X_4/(1 - X)$ if the picture is adopted that the $^3$He in the droplets moves rigidly attached to the $^4$He. Consequently the compensating upward flow rate of $^3$He is also $n_{31}$. It will be assumed that the counterflowing phases are in thermodynamic equilibrium at each cross-section. If we take an energy balance over a tube slice between $l$ and $l + dl$ we have to add the proper gravitational terms. If $-z$ is the vertical projection of $l$ and $g$ the acceleration due to gravity the energy balance reads

$$\dot{n}_4 \ m_4 \ d(\mu_4 + gz) + \dot{n}_{31} \ m_3 \ d(\mu_3 + gz + T_{sd})$$

$$-\dot{n}_{31} \ m_3 \ d(\mu_{3e} + gz + T_{s3e}) - \frac{\pi d^2}{4} \ d \left[ \kappa(T) \frac{dT}{dl} \right] - \dot{Q} \ dl = 0,$$  \hspace{1cm} (1)

where $\mu_4$, $\mu_3$ and $\mu_{3e}$ are the chemical potentials (free enthalpies) of $^4$He, dilute $^3$He and pure $^3$He, respectively; where $s_d$ is the entropy carried by the dilute mixture per unit mass of $^3$He in the droplet; $s_{3e}$ is the entropy of the pure $^3$He per unit mass (see also I); $\kappa(T)$ is the heat conductivity coefficient for the liquid in the tube, i.e. a weighted average of $\kappa_d$ and $\kappa_e$; $m_4(m_3)$ is the
molar weight of $^{4}\text{He}(^{3}\text{He})$; $\dot{Q}dl$ is the external heat supply per second and $T$ is the temperature. Values of $s_0(T)$ and $s_{3e}(T)$ are given in table I; note that the dependencies of $s_d$ and $s_{3e}$ on the pressure $p$ will not be taken into account.

If we make use of thermodynamic equilibrium, i.e. $\mu_3 = \mu_{3e}$ and put $\dot{Q} = 0$, eq. (1) reduces to

$$\dot{n}_d m_4 d (\mu_4 + gz) + \dot{n}_{3i} m_3 d \left[T (s_d - s_{3e})\right]$$

$$- \frac{\pi d^2}{4} k d \left(\frac{1}{T} \frac{dT}{dl}\right) = 0,$$

(2)

where $k = \kappa T$ is the appropriate average between $k_d$ and $k_c$ (see table I; for not too high flow rates we can use $k \approx k_c$). If $\dot{n}_4$ is smaller than some critical $\dot{n}_4$ value there is no superfluid contact between the successive droplets, such that $d(\mu_4 + gz)$ will be different from zero. The picture will be adopted in which droplets at $l = L_0$ have zero vertical velocity components; due to the mass density difference between droplets and surrounding $^3\text{He}$ the droplets will accelerate until a situation is reached in which the gravitational force acting on the droplet equals the frictional force. The details of this acceleration process will not be discussed, as relevant experimental information concerning the size and the individual velocities of the droplets is not available. In what follows we will assume that the dissipation due to a fall of a droplet, accompanied by upwards moving $^3\text{He}$, is equal to the loss in gravitational energy. This implies that we will disregard any change in this dissipation mechanism due to hydrodynamic pressure differences. Incidentally, the kinetic energy gained by the droplets in their fall can easily be shown to be negligible in this connection.

We proceed by formally disentangling the energy flow rate $\dot{n}_4 m_4 d(\mu_4 + gz)$ into the above indicated dissipation term and some residual term by introducing a reduced pressure $p' = p + \varrho_{3e} gz$, where $\varrho_{3e}$ is the mass density of pure $^3\text{He}$. In the absence of any droplet flow, $p'(z)$ would be constant, the fluid being pure $^3\text{He}$ in that case. Since the $p$ dependent part of $\mu_4(p, T, X)$ can be written as $p\varrho_4$, it follows immediately that

$$\dot{n}_4 m_4 d [\mu_4(p, T, X) + gz]$$

$$= \dot{n}_4 m_4 d \mu_4(p', T, X) + \dot{n}_4 m_4 (1 - \varrho_4 \varrho_{3e}) g dz,$$

(3)

where $\varrho_4$ is the $^4\text{He}$ partial volume in the droplets. The second term on the right-hand side of eq. (3) is precisely the gravitational energy loss rate mentioned above. It would vanish if the two densities $1/\varrho_4$ and $\varrho_{3e}$ were equal. From table I it follows, however, that $(1 - \varrho_4 \varrho_{3e}) = 0.435.
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The first term on the right-hand side of eq.(3) contains the quantity $p'$. As mentioned above, hydrodynamic pressure gradients in $p'$ will be disregarded; for that reason we can as well put $d\mu_4(p', T, X) = d\mu_4(0, T, X)$ in eq.(3). However, a correct evaluation of $\mu_4(p', T, X)$ should have given among other things, the exact energy dissipation rate. It can be proved in the Fermi gas model applicable to $^3$He, using the Gibbs–Duhem relation \(^1\)\(^2\), that

$$d\mu_4(0, T, X) \equiv -\frac{a_2}{3m_4} X d(T^2). \quad (4)$$

When the expressions for $s_d(T)$ and $s_3c(T)$ from table I are applied, eq. (2) is transformed into the differential equation

$$n_{31} \left(3a_2 - a_1\right) \frac{d}{dl} (T^2) + n_4 m_4 (1 - v_4 e_{3d}) g \frac{dz}{dl}$$

$$- \frac{\pi d^2}{4} k \frac{d}{dl} \left( \frac{1}{T} \frac{dT}{dl} \right) = 0. \quad (5)$$

If $\delta$ is the angle between the tube and the vertical direction, we have $dz/dl = -\cos \delta$. At given $n_4$ the gravitational energy loss rate can thus be reduced by taking $\delta \neq 0$. It should be realized, that the driving force on the droplets in the tube direction diminishes with a factor of $\cos \delta$. As this very probably leads to smaller critical $n_4$ values the proposition to take $\delta = 0$ seems to be less attractive. There is only scarce experimental evidence \(^7\) concerning the performance of superfluid injection refrigerators with non-vertical tubes.

It is useful to compare eq. (5) with the corresponding differential equation valid for the exhaust tube in a conventional dilution refrigerator. We recall equation (I,13) for that reason. It reads

$$\frac{n_3 a_2}{2} \frac{d}{dl} (T^2) - \left(\frac{n_3 m_3}{\rho_{3d}}\right)^2 \frac{128}{\pi d_1^4 n_d T^2} \frac{1}{T^2}$$

$$- \frac{\pi d_1^2}{4} k_d \frac{d}{dl} \left( \frac{1}{T} \frac{dT}{dl} \right) = 0. \quad (6)$$

Here $n_3$ is the $^3$He flow rate, $\rho_{3d}$ is the density of $^3$He with which it is present in the dilute mixture, and $n_d/T^2$ is the viscosity coefficient for dilute $^3$He; $d_1$ is the diameter of the exhaust tube. For equal flow rates $n_{31}$ and $n_3$ there is a slight difference between the convective (first) terms in eqs (5) and (6), the ratio $2 (2a_2/3 - a_1)/a_2$ being equal to 8/9. The dissipative (second) terms differ considerably. In equation (6) this term shows a proportionality with $n_3^2$, as it
should, while the term in eq. (5) is only proportional to \( n_4 \) or \( n_{31} \). It should be realized, however, that the gravitational force acting on a droplet balances a frictional force term, which in itself is proportional to \( n_{31} \). This implies the required proportionality with \( n_{31}^2 \). Finally the ratio of the conductive energy rates in eqs (5) and (6) equals \( k/k_d \), being slightly larger than one.

For the sake of comparison with the exhaust tube case it makes sense to transform the second-order differential equation (5) for \( T(l) \) into the following set of two first-order differential equations:

\[
\frac{df_s(t)}{dt} = \frac{2 [1 - f_s(t)]}{t} - \frac{1}{t^5 f_s(t)},
\]

(7)

\[
\frac{d\lambda}{dt} = \frac{1}{t^3 f_s(t)},
\]

(8)

where we have introduced a dimensionless temperature \( t = T/T_s \), and a dimensionless length parameter \( \lambda = l/l_s \), with

\[
T_s = \left[ \frac{\pi g (1 - v_4 \varepsilon_{3e}) k m_4 (1 - X)}{4X (\frac{3}{2}a_2 - a_1)^2} \right]^{1/4} \frac{d^{1/2}}{n_{31}^{1/4}} \cos^{1/4} \delta
\]

(9)

and

\[
l_s = \left( \frac{\pi k (1 - X)}{4g (1 - v_4 \varepsilon_{3e}) m_4 X} \right)^{1/2} \frac{d}{n_{31}^{1/2}} \frac{1}{\cos^{1/2} \delta}.
\]

(10)

The function \( f_s(t) \) is defined by

\[
\frac{\pi d^2}{4} k \frac{1}{T} \frac{dT}{dl} = f_s(t) \frac{n_{31}}{\delta a_2 - a_1} T^2.
\]

(11)

It can be given the following physical interpretation: \( f_s(t) \) expresses the heat flow rate due to heat conduction in terms of the convective energy flow rate in the tube (which is proportional to \( T^2 \), see eq. (5)). Note that the only difference between eq. (7) on the one hand and eq. (1,14) on the other is the occurrence of a fifth power of \( t \) instead of a seventh power.

Equations (7) and (8) are equivalent to the differential equations established by Pennings et al. \(^7\) for this case. These authors followed a numerical procedure in order to solve the equations. It can easily be shown, however, that the equations can be solved exactly. If we put \( x = t^2 [1 - f_s(t)] \) the solutions of eq. (7) read

\[
t^{-2} \exp (-x^2) - \pi^{-1} \text{erf} (-x) = \beta_1,
\]

(12)
while the solutions of eq. (8) read
\[ \lambda = x + \beta_2. \] (13)

Here \( \beta_1 \) and \( \beta_2 \) are two integration constants. The solution (12) is arrived at after the consecutive substitutions \( x = t^2 \left[ 1 - f_s(t) \right] \), \( t_1 = \ln t \), \( y = 2t_1 + x^2 \), leading to a separable differential equation. It is easily verified that differentiation of the expression (13) with respect to \( t \) yields eq. (8). A set of solutions for \( f_s(t) \) is given in fig. 3. When they are compared with the functions \( f_s(t) \) that

![Graphical representation of solutions of the differential equation](image)

Fig. 3. Graphical representation of solutions of the differential equation
\[ \frac{df_s(t)}{dt} = \frac{2 \left[ 1 - f_s(t) \right]}{t} - \frac{1}{t^5 f_s(t)} \]

for positive \( f_s \) and \( t \). The asymptotic curve is indicated by the symbol \( A \). The intersection of \( A \) with the line \( f_s = 1.75 \) is seen to give \( t = 0.626 \).

play a similar role in the case of the exhaust tube in a conventional dilution refrigerator (see fig. (1, 5). the resemblance is striking. For determining the integration constants \( \beta_1 \) and \( \beta_2 \) two additional conditions are needed. In the next section we deal with the question of the proper boundary relations connected with the two first-order differential equations (7) and (8). We close this section by noting that solutions (12) and (13) enable us to give an explicit analytical expression for \( T(l) \):

\[ T(l) = T_s \left[ \exp \left\{ \frac{- (\beta_2 - l/L)^2}{\beta_1 + \pi^2 \text{ erf} (\beta_2 - l/L)} \right\} \right]^{1/2}. \] (14)
3. Additional conditions for the function \( T(l) \)

The natural way to find additional conditions for \( T(l) \) is to specify the thermodynamic situation in chambers \( M \) and \( M_1 \), respectively. This leads to a trivial problem in the case of fig. 1, where only \(^4\)He is circulated: the temperature \( T_{M_1} \) will be of the order of 0.5 K, implying that our analysis of the droplet flow in the tube is certainly not valid in the close neighbourhood of \( M_1 \). It would have been valid only if none of the simplifying assumptions about the temperature dependence of the relevant physical quantities had been made. On the other hand, a complete treatment without these simplifications is in principle possible\(^9\), although explicit solutions of the differential equations can then no longer be found. We restrict ourselves to the analysis of the case of fig. 2, in which, due to the external \(^3\)He circulation, the temperatures \( T_{M_1} \) are generally sufficiently low to justify the use of the above-mentioned simplifying assumptions. Temperature distributions \( T(l) \) obtained for this case may partially (for the tube part close to \( M \)) serve as temperature distributions in the case of fig. 1.

A first condition can be derived at \( l = L_0 \), that is at the mixing chamber \( M \) (see fig. 1). The energy balance taken over chamber \( M \) reads

\[
\dot{n}_4 m_4 [(\mu_4 + gZ)_{\text{inlet}} - (\mu_4 + gZ)_{\text{outlet}}] - \dot{n}_{31} m_3 [(\mu_3 + gZ + T_M s_3)_{\text{outlet}} - (\mu_3 + gZ + T_M s_3)_{\text{outlet}}] + \frac{\pi d^2}{4} k \frac{1}{T_M} \left( \frac{dT}{dl} \right)_M + \dot{q}_M = 0, \tag{15}
\]

where \( \dot{q}_M \) is an externally applied heat load (by definition equal to the cooling power). In equation (15) it is assumed that the temperature in \( M \) is uniform (equal to \( T_M \)). If in addition a uniform pressure \( p_M \) is assumed and if it is furthermore assumed that the \(^3\)He molar concentration of all droplets in \( M \) is equal to \( X \) (see, however, ref. 12 for the profound effects on superfluid injection cooling if the \(^3\)He concentration at the \(^4\)He inlet is lower) then eq. (15) reduces to

\[
\dot{q}_M = \dot{n}_{31} (a_2 - a_1) T_M^2 - \frac{\pi d^2}{4} k \frac{1}{T_M} \left( \frac{dT}{dl} \right)_M. \tag{16}
\]

Here the \( z \) coordinates of inlet and outlet have been taken equal. In view of the definition of \( f_s(t) \) given in eq. (11) the energy balance equation (16) can alternatively be written as

\[
f_s \left( \frac{T_M}{T_s} \right) = 1 + \frac{\frac{1}{3} \dot{n}_{31} a_2 T_M^2 - \dot{q}_M}{\dot{n}_{31} (3a_2 - a_1) T_M^2}. \tag{17}
\]
For $\dot{q}_M = 0$ the temperature $T_M$ will be lowest; from eq. (17) it is observed that in that case $f_s(T_M/T_s) = 1.75$.

With respect to the condition to be derived at $l = L_1$, where $L_1 - L_0$ is the length of the tube between $M$ and $M_1$, we have to determine the total energy flow rate entering $M_1$ due to the application of the $^4$He injection process in $M$. Taking into account that the incoming superfluid $^4$He in $M_1$ leaves this chamber again through the $^3$He exhaust tube at the same temperature $T_{M_1}$, we find that $\dot{q}_M(0, T_{M_1}, \chi) = 0$ (see eq. (4)). Therefore, the heat load on $M_1$ due to the droplet flow $\dot{q}_{M_1}$ is given by

$$\dot{q}_{M_1} = n_{31}(a_2 - a_1) T_{M_1}^2 - \frac{\pi d^2}{4} k \left[ \frac{1}{T_{M_1}} - \frac{dT}{dl} \right]_{M_1}, \quad (18)$$

assuming exactly the form of eq. (16). In terms of the function $f_s(t)$ condition (18) reads alternatively

$$f_s \left( \frac{T_{M_1}}{T_s} \right) = 1 + \frac{\frac{1}{2} n_{31} a_2 T_{M_1}^2 - \dot{q}_{M_1}}{n_{31} (\frac{3}{2} a_2 - a_1) T_{M_1}^2}, \quad (19)$$

which, as a matter of course, assumes the form of eq. (17).

The two conditions (16) and (18) determine the function $T(l)$ of eq. (14) completely, if apart from the parameters $n_{31}, d, \delta$ and $L$ also the quantities $\dot{q}_M$ and $\dot{q}_{M_1}$ are known.

In deriving temperature distributions $T(l)$ we will follow a procedure which starts from results obtained in I concerning the cooling power in the case of continuous dilution refrigeration. It is useful in this respect to recall the energy balance equation on $M_1$ for the case in which only $^3$He is circulated (eq. (I, 34)).

$$\dot{n}_3 T_{M_1}^2 \left[ a_1 \left( 1 - \frac{f_d(T_{M_1}/T_c)}{2} \right) - a_2 \left( 1 - \frac{f_d(T_{M_1}/T_d)}{2} \right) \right] + \dot{Q}_{M_1} = 0. \quad (20)$$

Here $-f_c(t)$ expresses the heat flow rate due to heat conduction in the $^3$He supply tube in terms of the temperature-dependent part of the $^3$He enthalpy flow rate, while $f_d(t)$ is defined similarly in the case of the $^3$He exhaust tube (see eqs (I, 33) and (I, 18); the reference temperatures $T_c$ and $T_d$ are given in eqs (I, 31) and (I, 16)); $\dot{Q}_{M_1}$ is the heat load on chamber $M_1$, being the cooling power of chamber $M_1$ in the conventional $^3$He circulating dilution refrigerator.

Now suppose we add to $M_1$ the tube connecting $M_1$ and $M$ in such a way that the situation in fig. 2 is obtained. As we do not intend to put an external heat load on $M_1$ in this case, the natural way to adapt the balance equation
(20) to the case of both \(^3\)He and \(^4\)He circulation is to identify the quantity \(Q_{M_1}(T_{M_1})\) in eq. (20) with \(q_{M_1}(T_{M_1})\) given in eq. (18). Fortunately, the function \(Q_{M_1}(T_{M_1})\) has already been calculated in I for a number of cases. This leads therefore to the following procedure: Let \(n_4, d, \delta, T_{M_1}\) be given. To the temperature value \(T_{M_1}\) corresponds a given value \(Q_{M_1}\) (calculated in I for some given set-up). Putting \(g_{M_1} = Q_{M_1}\) and applying eq. (19) yields \(f_s(T_{M_1}/T_s)\). The constants \(\beta_1\) and \(\beta_2\) can now be obtained from eqs (12) and (13) by using \(\lambda = L_1/l_s\) and inserting \(t = T_{M_1}/T_s\). It is then possible to calculate \(T(l)\) for \(l < L_1\). In doing so it is necessary to check whether condition (17) can be satisfied with positive \(g_{M_1}\). The easiest way is to calculate \(j_s(T(l)/T_{M_1})\) first, using eq. (13), and then insert it in eq. (17) in order to determine \(g_{M_1}\). The value \(l = L_0\) for which \(q_{M_1} = 0\) leads to a tube length \(L = L_1 - L_0\) with which the lowest temperature \(T_M\) can be obtained.

The above procedure is correct only if the additional \(^4\)He flow entering \(M_1\) and leaving it through the \(^3\)He exhaust tube does not change the derived \(f_s(t)\) function. It can be deduced from the derivation in I that the actual form of \(f_s(t)\) will not change as long as \(d(\mu_4 + gz) = 0\) is valid in the exhaust tube. This will probably only be the case for relatively low values of the \(^4\)He flow rate \(n_4\). It can be seen, however, that there is another, very important condition under which the function \(f_s(t)\) will remain the same: using the Gibbs–Duhem relation (eq. (1,8)) the energy balance equation for the exhaust tube in the case of an additional \(^4\)He flow rate \(n_4\) can be shown to contain the term \((\dot{n}_4 m_4 - \dot{n}_3 m_4 (1 - X)/X) d(\mu_4 + gz)\). If therefore \(\dot{n}_3\) is chosen equal to \(\dot{n}_3\) this term vanishes although \(d(\mu_4 + gz)\) may differ from zero and the function \(f_s(t)\) derived in I will not be influenced. The \(^3\)He and both the superfluid and normal component of the \(^4\)He move with equal velocity in the exhaust tube.

In figure 4 a number of temperature distributions \(T(l)\) are given, calculated in the way described above, starting from cooling power results obtained in I for the case of continuous dilution refrigeration with \(^3\)He circulation. The various tube lengths are taken such that the cooling power \(q_M\) is almost zero for a mixing chamber \(M\) placed at the very end. For the reason mentioned above, we used \(\dot{n}_3\) values equal to the external \(^3\)He flow rate values \(\dot{n}_3\) taken in I. In that case a situation arises in which the demixing and mixing in chamber \(M_1\) are suppressed and the chamber only acts as a heat exchanger.

An interesting problem concerns the cooling power of a double circulation dilution refrigerator in which, apart from \(d, \dot{n}_3, \delta\) and \(L\) the tube length, too, is a given quantity. This cooling power function can be obtained by drawing in fig. 4 the line \(l = L\) and by calculating \(q_M\) at each of the points of intersection with the \(T(l)\) curves. A typical result is shown in fig. 5.
On $^3$He-$^4$He dilution refrigerators

<table>
<thead>
<tr>
<th>curve</th>
<th>L [mm]</th>
<th>d [mm]</th>
<th>$\cos \delta$ (mol s$^{-1}$)</th>
<th>$\dot{q}_M$ [J s$^{-1}$ x 10$^3$]</th>
<th>$T_{M_1}$ [mK]</th>
<th>$T_M$ [mK]</th>
<th>$T_{min}$ eq.(21) [mK]</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>67</td>
<td>2.6</td>
<td>1</td>
<td>45.17</td>
<td>28.24</td>
<td>0.13</td>
<td>4.28</td>
</tr>
<tr>
<td>b</td>
<td>57</td>
<td>2.6</td>
<td>1</td>
<td>41.21</td>
<td>27.50</td>
<td>0.13</td>
<td>4.26</td>
</tr>
<tr>
<td>c</td>
<td>52</td>
<td>2.6</td>
<td>1</td>
<td>39.23</td>
<td>27.12</td>
<td>0.27</td>
<td>4.40</td>
</tr>
<tr>
<td>d</td>
<td>35</td>
<td>2.6</td>
<td>1</td>
<td>32.26</td>
<td>25.75</td>
<td>0.19</td>
<td>4.34</td>
</tr>
<tr>
<td>e</td>
<td>30</td>
<td>2.6</td>
<td>1</td>
<td>30.26</td>
<td>25.34</td>
<td>0.22</td>
<td>4.36</td>
</tr>
</tbody>
</table>

Fig. 4. Temperature distributions along the tube $P$ connecting the chambers $M$ and $M_1$ (see fig. 2) for some choices of the parameters set $L$, $d$, $\delta$, $\dot{q}_{31}$, $Q_{M_1}$, $T_{M_1}$. In all cases one additional mm for $L$ leads to negative $\dot{q}_M$ values. The calculated cooling powers $\dot{q}_M$ as well as the temperatures $T_M$ are given. For comparison $T_{min}$ of eq. (21) has been given as well.

The starting values $Q_{M_1}$ and $T_{M_1}$ in the respective cases a to e correspond to cooling powers as calculated in I for the case of $^3$He circulation only. For that reason these cooling power results of I are recalled in fig. 5 (curve 1). The choices for the starting value in the cases 4a to 4e coincide with the dots in the $(Q_{M_1}, T_{M_1})$ plane of fig. 5.

4. Minimum temperature

It can be seen from fig. 3 that all $f_s(t)$ curves for small $t$ merge into the same asymptotic curve $A$. The lowest $t$ values are obtained if the condition $\dot{q}_M = 0$ is met. On the one hand we know that $f_s(T_{M}/T_s) = 1.75$ holds in that case; on the other hand it can be deduced that the $f_s$ curves relevant in superfluid injection cooling practically coincide with the asymptotic curve $A$ at and below $t = 1$. The latter statement is a consequence of considerations concerning realistic tube length and flow rate values $L$ and $\dot{q}_s$; this may be compared with the analogous observation in I. With regard to the minimum temperature to be obtained in realistic cases it turns out therefore that the intersection point of curve $A$ with the line $f_s = 1.75$ is of interest. The value of $t$ belonging to this point is equal to 0.626. In practice the minimum temperature to be obtained is therefore

$$T_{min} = 0.626 T_s.$$  (21)

This expression agrees with that of Pennings et al. 7). These authors arrived
Fig. 5. Curve I: Cooling power as a function of temperature $\dot{Q}_{M1}(T_{M1})$ as calculated in I for the case of $^3$He circulation only. Values for the diameter of the $^3$He supply tube $d_a$, the $^3$He exhaust tube $d_1$ and the $^3$He flow rate $\dot{n}_3$ are given for completeness. The dots refer to the starting values $\dot{Q}_{M1}$ and $T_{M1}$ as used in the calculation of the temperature distributions along tube P (see fig. 4).

Curve A: Cooling power as a function of temperature $\dot{Q}_{M}(T_M)$ for a choice of the parameter set $L$, $d$, $\delta$, $\dot{n}_{31}$. The function $\dot{Q}_{M}(T_M)$ has been calculated by requiring that the initial $\dot{Q}_{M1}$ and $T_{M1}$ values lie on curve 1. Note that choice of larger $\dot{Q}_{M1}$ and $T_{M1}$ values, keeping $d$, $\delta$ and $\dot{n}_{31}$ fixed, allows for larger $L$ values.

at the minimum-temperature expression by means of numerical integration of differential equations related to those established by us. It is emphasized that for relatively small tube lengths $L$ and flow rates $\dot{n}_4$ the lowest temperature to be reached in chamber M will be higher than indicated in eq. (21).

The reference temperature $T_s$ in eq. (21) should be chosen as low as possible. If actual values are taken for the quantities $k$, $m_4$, $g$, $v_4$, $\ell_{3c}$, $X$, $a_2$ and $a_1$, eq. (21) for the minimum attainable temperature in the thusfar described superfluid injection methods can be written as

$$T_{min} = 8.14 \times 10^{-3} \frac{d^{1/2}}{\dot{n}_{31}^{1/4}},$$

where $T_{min}$ is in kelvins, $d$ in meters and $\dot{n}_{31}$ in moles per second; $\delta$ has been taken equal to zero. It is well known, that at given $d$ a critical $\dot{n}_{31}$ (or $\dot{n}_4$) value
exists above which no refrigeration can be obtained. The droplets are in contact with each other then, prohibiting the upward flow of concentrated $^3\text{He}$. Pennings et al. 9), using a model in which the droplets are assumed to be spherical, derived the expression

$$\dot{n}_{4,\text{crit}} \approx 4.6 \times 10^5 d^{10/3}. \quad (23)$$

Inserting (23) in (22) we can obtain a minimum-temperature expression, either in terms of critical flow rates or in terms of “critically used” tube diameters $d_{\text{crit}}$, reading

$$T_{\text{min}} \approx 1.7 \times 10^{-3} \frac{1}{\dot{n}_{31,\text{crit}}^{1/10}} \approx 6.1 \times 10^{-4} \frac{1}{d_{\text{crit}}^{1/3}}. \quad (24)$$

Note the (relatively small) difference between the number 6.1 and the corresponding number 6.3 in the expression obtained in ref. 9.

It is useful to compare these minimum-temperature expressions (24) with that obtained in I for the case of single cycle dilution refrigeration ($^3\text{He}$ exhaustion only). If we recall this expression (see I,27) and insert the actual values for the quantities $n_d$, $k_d$, $m_3$, $a_2$ and $\varrho_{3d}$, this expression reads

$$T_{\text{min}} \quad (^3\text{He circ.}) = 3.9 \times 10^{-4} \frac{1}{d_1^{1/3}} \quad (25)$$

It is emphasized that minimum temperatures to be obtained in a continuously operating $^3\text{He}$ circulating dilution refrigerator will be higher. If, however, high quality heat exchangers can be applied, the minimum temperature to be reached in these $^3\text{He}$ circulating refrigerators will be only slightly larger than indicated by eq. (25) (see the discussion in I on this point). In such a case the minimum temperature to be reached at given $d_1$ will be about equal to — or even somewhat smaller than — the minimum temperature to be obtained in a superfluid injection dilution refrigerator in which $d_{\text{crit}} = d_1$, (compare the numbers 3.9 and 6.1 occurring in eqs (25) and (24)). In the $^3\text{He}$ circulation case this minimum temperature can be reached independent of the value of $\dot{n}_3$ (however, for relatively small $\dot{n}_3$ values this will necessitate the use of a relatively long exhaust tube). In the case of superfluid injection and the “critical” use of the tube implies a definite $\dot{n}_{31}$ value equal to $X/(1 - X) \dot{n}_{4,\text{crit}}$. Enlarging the tube diameters in both cases may lead to lower minimum temperatures; in the $^3\text{He}$ circulating case this will generally ask for a longer exhaust tube, unless the flow rate is sufficiently enlarged; in the superfluid injection case this asks for an increase in $\dot{n}_4$ proportional to $d^{10/3}$ (see eq. (23)) and a corresponding decrease of tubelength proportional to $d^{-2/3}$ (apply eqs (23) and (10)).

On $^3\text{He}-^4\text{He}$ dilution refrigerators
physical reason for the differences between the two cases finds its origin in the fact that the dissipation in the case of droplet flow is independent of the tube diameter, while it is proportional to $1/d^4$ in the case of an exhaust tube.

An advantage of the superfluid injection method is that the $^4\text{He}$ can be circulated by means of a thermomechanical $^4\text{He}$ pump, which is relatively easy to handle. Flow rates $\dot{n}_4$ equal to $1.4 \times 10^{-3}$ moles per second have been realized \(^1\) leading to a minimum temperature of about 4 mK, which is in reasonable agreement with the value predicted from eqs (22) or (24). In view of these remarks, the existing superfluid injection methods offer a promising alternative, not necessarily because of the lowest temperatures to be reached, but mainly because of the fact that heat exchange problems as met in $^3\text{He}$ circulating methods have been circumvented.

Philips Research Laboratories · Eindhoven, February 1979

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