Abstract

This paper describes a number of constructions of binary Linear Unequal Error Protection (LUEP) codes. The separation vectors of the constructed codes include those of all optimal binary LUEP codes of length less than or equal to 15.

AMS: 94B05, 94B60.

1. Introduction

Consider a binary linear code \( C \) of length \( n \) and dimension \( k \) with generator matrix \( G \) to be used on a binary symmetric channel. In many applications it is necessary to provide different protection levels for different components \( m_i \) of the input message word \( m \). For example in transmitting numerical (binary) data, errors in the more significant bits are more serious than are errors in the less significant bits, and therefore more significant bits should have more protection than less significant bits.

A suitable measure for these protection levels for separate positions in input message words is the separation vector \(^1\).

Definition

For a binary linear \([n,k]\) code \( C \) the separation vector \( s(G) = (s(G)_1, s(G)_2, \ldots, s(G)_k) \) with respect to a generator matrix \( G \) of \( C \) is defined by

\[
s(G)_i := \min \{ \text{wt}(m G) | m \in \{0, 1\}^k, m_i = 1 \},
\]

where \( \text{wt}(.) \) denotes the Hamming weight function.

This separation vector \( s(G) \) guarantees the correct interpretation of the \( i^{th} \) message bit whenever Nearest Neighbour Decoding (ref. 2 p. 11) is applied and no more than \((s(G)_i - 1)/2\) errors have occurred in the transmitted code-word \(^1\).

A linear code that has a generator matrix \( G \) such that the components of the corresponding separation vector \( s(G) \) are not mutually equal is called a Linear...
Unequal Error Protection (LUEP) code. By permuting the rows of a generator matrix \( G \) we may obtain a generator matrix \( G' \) for the code such that \( s(G') \) is nonincreasing, i.e. \( s(G'_i) \geq s(G'_{i+1}) \) for \( i = 1, 2, \ldots, k - 1 \). In this paper we always assume that the rows in generator matrices are so ordered that the corresponding separation vectors are nonincreasing.

Any LUEP code \( C \) has a so-called optimal generator matrix \( G^* \). This means that the separation vector \( s(G^*) \) is componentwise larger than or equal to the separation vector \( s(G) \) of any generator matrix \( G \) of \( C \), denoted by \( s(G^*) \geq s(G) \) (\( x \geq y \) means \( x_i \geq y_i \) for all \( i \)). The vector \( s = s(G^*) \) is called the separation vector of the linear code \( C \). We use the notation \([n,k,s]\) for \( C \).

For any \( k \in \mathbb{N} \) and \( s \in \mathbb{N}^k \) we define \( n(s) \) to be the length of the shortest binary linear code of dimension \( k \) with a separation vector of at least \( s \), and \( n^e(s) \) to be the length of the shortest binary linear code of dimension \( k \) with separation vector (exactly) \( s^* \). An \([n(s),k,s]\) code is called length-optimal. It is called optimal if an \([n(s),k,t]\) code with \( t \geq s \), \( t \neq s \) does not exist. In refs 3 and 4 a number of bounds for the functions \( n(s) \) and \( n^e(s) \) are derived. In ref. 5 methods for constructing LUEP codes from shorter codes are described.

In refs 3 and 4 an incomplete list of the separation vectors of the optimal binary LUEP codes of length less than or equal to 15 is given. In this paper we provide the complete list of the separation vectors of all optimal binary LUEP codes of length less than or equal to 15, together with examples of generator matrices having these separation vectors. Furthermore, we give a number of constructions of infinite series optimal binary LUEP codes.

2. Constructions

Table I provides the separation vectors of all optimal binary LUEP codes of length less than or equal to 15. In this table, \( n \) denotes the length of the code, \( k \) denotes the dimension, and \( d(n,k) \) denotes the maximal minimum distance of a binary code of length \( n \) and dimension \( k \). The brackets and commas commonly appearing in separation vectors have been deleted. Only in the cases where a component of a separation vector is larger than 9, it is followed by a point (.). Examples of codes having the parameters given in table I are constructed below. The bounds in ref. 4 can be used to show that certain LUEP codes are optimal. They are also useful in showing that table I is complete. In cases where these bounds did not work, methods of exhaustive search were used to show that codes with certain parameters do not exist. Table I is the same table as table I in ref. 4, extended by the parameters \([14,10,(433332222222)], [15,3,(994)]\), \([15,8,(733333333)], [15,8,(555544443)], [15,8,(555444444)]\) and \([15,11,(433332222222)]\). In (ref. 4 table I) no references to constructions were given, which has been done in this paper.
Some constructions of optimal binary linear unequal error protection codes

TABLE I

The separation vector of all binary optimal LUEP codes of length less than or equal to 15.

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
<th>d(n,k)</th>
<th>separation vector</th>
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<td>4</td>
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<td>13</td>
<td>2</td>
<td>$A\ 322222222222$</td>
</tr>
</tbody>
</table>
Some constructions of optimal binary linear unequal error protection codes

In this paper we frequently use two results of ref. 4. Hence we repeat these results in the following two theorems.

Theorem 1 (ref. 4, theorem 12)

For any \( k \in \mathbb{N} \) and nonincreasing \( s \in \mathbb{N}^k \) we have that

\[
n^{ex}(s_1, \ldots, s_k) \geq s_i + n(\delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_k)
\]

holds for any \( i \in \{1, \ldots, k\} \), where

\[
\delta_j := \begin{cases} 
    s_j - \lfloor s_j/2 \rfloor & \text{for } j < i \\
    \lfloor s_j/2 \rfloor & \text{for } j > i,
\end{cases}
\]

(where \( \lfloor x \rfloor \) denotes the largest integer smaller than or equal to \( x \), and \( [x] \) denotes the smallest integer larger than or equal to \( x \)).

Theorem 2 (ref. 4, corollary 14)

For any \( k \in \mathbb{N} \) and any nonincreasing \( s \in \mathbb{N}^k \), the function \( n(s) \) satisfies the following inequalities,

a. \( n(s_1, s_2, \ldots, s_k) \geq s_1 + n(\lfloor s_2/2 \rfloor, \ldots, \lfloor s_k/2 \rfloor) \),

b. \( n(s_1, s_2, \ldots, s_k) \geq \sum_{i=1}^{k} \lfloor s_i/2^{i-1} \rfloor \).

Construction A

For \( n, k \in \mathbb{N}, n \geq k + 1 \), the \( k \) by \( n \) matrix

\[
\begin{bmatrix}
I_k & 11111\ldots111 \\
1 & 0_{k-1,n-k-1} \\
\vdots & \vdots \\
1 & \end{bmatrix}
\]

(1)

is a generator matrix of an optimal binary \([n,k,(n-k+1,2,2,\ldots,2)]\) code (\( I_k \) denotes the identity matrix of order \( k \), \( 0_{k-1,n-k-1} \) denotes the all-zero \( k - 1 \) by \( n - k - 1 \) matrix).

Proof

It is easy to check that the parameters of the code are correct. Furthermore by theorem 2b the length of a \( k \)-dimensional binary code with separation vector \((n-k+1,2,2,\ldots,2)\) is at least \( n \), and with separation vector larger than \((n-k+1,2,2,\ldots,2)\) is at least \( n + 1 \) (by \( s \geq t \) \((s \text{ larger than } t)\) we mean \( s \geq t, s \neq t \)).
Construction B

For \( k \in \mathbb{N}, k \geq 4 \), the \( k \) by \( 2k - 1 \) matrix

\[
\begin{bmatrix}
00 \ldots 0 & 11 \ldots 1 & 0 \\
I_{k-1} & I_{k-1} & \vdots \\
& & 1 \\
& & & 1 \\
& & & \vdots \\
& & & 1
\end{bmatrix}
\]

(2)

is a generator matrix of an optimal binary \([2k - 1, k, (k - 1, 3, 3, \ldots, 3)]\) code.

Proof

It is easy to verify that the parameters of the code are correct. By theorem 2b, we have that the length of a \( k \)-dimensional binary code with separation vector \((k - 1, 3, 3, \ldots, 3)\) is at least \( 2k - 1 \). Application of theorem 2b to a \( k \)-vector \( s \) with \( s_i \geq k \) and \( s_i \geq 3 \) for \( i = 2, \ldots, k \) shows that \( n(s) \geq 2k \). Application of the theorems 1 and 2 to a \( k \)-vector \( s \) such that \( s_1 = k - 1 \), \( s_2 \geq 4 \), \( s_i \geq 3 \) for \( i = 3, \ldots, k - 1 \), and \( s_k = 3 \) shows that

\[
\begin{align*}
n^{ex}(s) & \geq 3 + n(s_1 - 1, \ldots, s_{k-1} - 1) \\
& \geq 3 + s_1 - 1 + n((s_2 - 1)/2, \ldots, (s_{k-1} - 1)/2) \\
& \geq 3 + k - 2 + n(2, 1, 1, \ldots, 1) \\
& \geq k - 2 \\
& \geq 3 + k - 2 + k - 1 = 2k.
\end{align*}
\]

Furthermore it is not difficult to check that a binary \([2k - 1, k, (k - 1, 4, 4, \ldots, 4)]\) code does not exist. Finally, by theorem 2b the length of a \( k \)-dimensional binary code with a separation vector of at least \((k - 1, 5, 4, 4, \ldots, 4)\) is at least \( 2k \). These observations show that the code in construction B is optimal.

Construction C

For \( n, k \in \mathbb{N}, n \geq \max\{2k, k + 4\} \), the \( k \) by \( n \) matrix

\[
\begin{bmatrix}
00 \ldots 0 & 11 \ldots 1 & 11 \ldots 1 & 10 \\
I_{k-1} & I_{k-1} & 0 & 11 \\
& & 11 \\
& & \vdots \\
& & 11
\end{bmatrix}
\]

(3)

is a generator matrix of an optimal binary \([n, k, (n - k, 4, 4, \ldots, 4)]\) code.
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Proof
Similar to the proof of construction A.

Construction D
For \( p, q \in \mathbb{N}, p \geq q \geq 2 \), the \( p + q + 2 \) by \( 2p + 3q + 3 \) matrix

\[
\begin{bmatrix}
00\ldots0 & 1110 & 11\ldots1 & 00\ldots0 & 11.1 \\
00\ldots0 & 1101 & 00\ldots0 & 11\ldots1 & 11.1 \\
0101 & 0101 & I_p & 0 & 0 \\
0101 & : & : & : & : \\
0101 & 1010 & 0 & I_q & 0 \\
1010 & : & : & : & : \\
1010 & 1010 & 0 & q-1 & 0
\end{bmatrix}
\]

is a generator matrix of an optimal binary \([2p + 3q + 3, p + q + 2, (p + q + 2, 2q + 2, 4, 4, \ldots, 4)]\) code.

Proof
Similar to the proof of construction A.

Construction E
For \( p, q, r \in \mathbb{N}, p \geq 3, r \geq 2, q \geq r - p + 2 \), the \( p \) by \( (2p + q + 2r - 4) \) matrix

\[
\begin{bmatrix}
11\ldots1 & 00\ldots0 & 11\ldots1 & 00\ldots0 & 11.1 \\
00\ldots0 & 00\ldots0 & 11\ldots1 & 11\ldots1 & 00\ldots0 \\
I_{p-2} & I_{p-2} & 1 & 1 & 0 \\
 & : & 1 & 1 & 0 \\
 & : & 0 & 0 & 0 \\
 & 1 & 1 & : & : \\
 & : & r & r & q
\end{bmatrix}
\]

is a generator matrix of an optimal binary \([2p + q + 2r - 4, p, (p + q + r - 2, 2r, 4, 4, \ldots, 4)]\) code.
Proof
Similar to the proof of construction A.

Construction F
For \( p, q \in \mathbb{N} \), \( p \geq 3 \), \( q \geq 2 \), \( q \geq p - 2 \), the \( p \) by \( p + 3q \) matrix
\[
\begin{bmatrix}
00...0 & 11...1 & 11...1 & 00...0 & 11...1 \\
00...0 & 11...1 & 00...0 & 11...1 & 11...1 \\
I_{p-2} & I_{p-2} & I_{p-2} & I_{p-2} & I_{p-2}
\end{bmatrix}
\]
\(
\begin{bmatrix}
1 & 1 \\
1 & 0 & & 1 & 0 & 0 \\
\vdots & \vdots \\
1 & 1 \\
\end{bmatrix}
\)
\[
q + 1 & q + 1 & q - (p - 2)
\]
(6)
is a generator matrix of an optimal binary \([p + 3q, p, (2q + 1, 2q + 1, 4, 4, \ldots, 4)]\) code.

Proof
Similar to the proof of construction A.

Construction G
For \( p \in \mathbb{N} \), the \( 2p \) by \( 4p \) matrix
\[
\begin{bmatrix}
00...0 & 1 1 1 0 & 11...1 & 00..0 \\
00...0 & 1 1 0 1 & 00..0 & 11...1 \\
I_{2p-2} & I_{2p-2} & I_{2p-2} & I_{2p-2}
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 0 1 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 0 1 0 \\
0 1 0 1 \\
\vdots & \vdots & \vdots & \vdots \\
0 1 0 1 \\
\end{bmatrix}
\]
(7)
is a generator matrix of a binary \([4p, 2p, (p + 2, p + 2, 4, 4, \ldots, 4)]\) code. For \( p = 2, 3 \) the codes are optimal, but in general they are not.

In ref. 6 the codes from construction G are treated extensively, the weight enumerators and automorphism groups are determined completely and a majority logic decoding method for these codes is given. For \( p = 3 \) we obtain a \([12, 6, (5, 5, 4, 4, 4, 4)]\) optimal LUEP code. By deleting the row and column pairs \((6, 4), (5, 3)\) and \((4, 2)\) successively we obtain \([11, 5, (5, 5, 4, 4, 4)], [10, 4, (5, 5, 4, 4)]\) and \([9, 3, (5, 5, 4)]\) optimal LUEP codes respectively.
Some constructions of optimal binary linear unequal error protection codes

Construction H
For \( p \in \mathbb{N} \), \( p \geq 3 \), the \((p + 2)\) by \((4p + 1)\) matrix

\[
\begin{bmatrix}
00\ldots0 & 11\ldots1 & 11\ldots1 & 00\ldots0 & 0 \\
00\ldots0 & 11\ldots1 & 00\ldots0 & 11\ldots1 & 0 \\
I_p & I_p & I_p & I_p & 1 \\
: & : & : & : & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(8)

is a generator matrix of a length-optimal binary \([4p + 1, p + 2,(2p,2p,5,5,\ldots,5)]\) LUEP code.

Proof
It is easy to check that the code has the given parameters. By theorem 2b the length of a \((p + 2)\)-dimensional binary code with separation vector \((2p,2p,5,5,\ldots,5)\) is at least \(4p + 1\).

For \( p = 3 \) this construction gives a \([13,5,(6,6,5,5,5)]\) optimal LUEP code. Furthermore table I refers to the following trivial constructions.

Construction I
For \( p, q \in \mathbb{N}, p > q \), the 2 by \((p + 2q)\) matrix

\[
\begin{bmatrix}
11\ldots1 & 00\ldots0 & 11\ldots1 \\
00\ldots0 & 11\ldots1 & 11\ldots1
\end{bmatrix}
\]

\[
\begin{array}{c c c}
p & q & q \\
\hline
1 & 0 & 0 \\
0 & 1 & 0 \\
\vdots & \vdots & \vdots \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

(9)

is a generator matrix of an optimal binary \([p + 2q,2,(p + q,2q)]\) LUEP code.

Construction J
If the matrix \( G_1 \) has separation vector \( s(G_1) \) such that \( s(G_1)_k \geq 2 \), then the matrix

\[
G_2 = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
G_1 \\
\end{bmatrix}
\]

(10)

has separation vector \( s(G_2) = (s(G_1),2) \).
Construction $K_i$

If the matrix $G_1$ has separation vector $s(G_1)$ then the matrix

$$G_2 := [G_1 \mid e_i], \quad (11)$$

where $e_i$ is the vector with a 1 on the $i^{th}$ position and zeros elsewhere, has separation vector $s(G_2) = s(G_1) + e_i$.

The following theorem can be used to determine whether construction $K_i$ gives an optimal code.

**Theorem 3**

If $s$ is such that for all $t \geq s$, $t \neq s$, it holds that $n(t) > n(s)$ and if $G$ is a generator matrix of a binary optimal $[r + n(s), k, (r, 2s)]$ code, then the code generated by $[G \mid e_1 \mid e_1 \ldots \mid e_1]$ is a binary optimal $[r + t + n(s), k, (r + t, 2s)]$ code for $t$ in $\mathbb{N}$ arbitrary.

**Proof**

Let $s$ and $G$ fulfill the conditions mentioned above. By theorem 2a we have that

a) $n(r + t, 2s) \geq r + t + n(s)$.

b) $n(r + t + 1, 2s) \geq r + t + 1 + n(s) > r + t + n(s)$.

c) $n(r + t, 2s + u) \geq r + t + n([s_1 + u_1/2], \ldots, [s_k + u_k/2]) \geq r + t + 1 + n(s)$

for $u \geq 0$, $u \neq 0$.

Combination of a), b) and c) shows that the code generated by $[G \mid e_1 \mid e_1 \ldots \mid e_1]$ is optimal.

**Construction L**

Adding an overall parity-check bit to a binary $[n, k, s = (s_1, \ldots, s_k)]$ code gives a binary $[n + 1, k, s' = (2[(s_1 + 1)/2], \ldots, 2[(s_k + 1)/2])]$ code.

Sporadic constructions referred to in table I are the following.

**Construction M**

The 7 by 14 matrix

$$\begin{bmatrix}
0001\; 1111\; 1000\; 0000 \\
0001\; 1000\; 0111\; 0000 \\
0001\; 0100\; 0100\; 0101 \\
0000\; 1010\; 0100\; 0011 \\
0110\; 0001\; 1000\; 0000 \\
0100\; 1000\; 0000\; 0100 \\
1000\; 0000\; 0000\; 1111
\end{bmatrix} \quad (12)$$
Some constructions of optimal binary linear unequal error protection codes

is a generator matrix of an optimal binary \([14,7,(5,5,5,5,4,4,4)]\) LUEP code. Deleting the first column and the last row from the matrix in (12) gives an optimal binary \([13,6,(5,5,5,5,4,4,4)]\) code. Deleting the first two columns and the last two rows from the matrix in (12) gives an optimal binary \([12,5,(5,5,5,5,4)]\) LUEP code.

Construction N

Application of \([5,\text{construction 1}]\) with \(m = 1, q = 2\) and \(G_1\), a generator matrix of the \([7,4,(3,3,3,3)]\) Hamming code gives an optimal binary \([14,5,(7,6,6,6,6)]\) LUEP code.

Construction O

The 6 by 15 matrix

\[
\begin{bmatrix}
000011111000111 \\
000000000111111 \\
100011000100100 \\
010010100010010 \\
001010010001001 \\
000100010001001
\end{bmatrix}
\]

(13)

is a generator matrix of an optimal binary \([15,6,(7,6,5,5,4)]\) LUEP code.

Construction P

The 7 by 15 matrix

\[
\begin{bmatrix}
000001111111000 \\
000001110000111 \\
100001001000100 \\
010001001000010 \\
001000100100001 \\
000100100010001 \\
000010100001001
\end{bmatrix}
\]

(14)

is a generator matrix of an optimal binary \([15,7,(7,6,4,4,4,4,4)]\) LUEP code.

Construction Q

By deleting the 8th column from the matrix in (14) we obtain a generator matrix of an optimal binary \([14,7,(6,5,4,4,4,4,4)]\) code.

Construction R

The 8 by 15 matrix
where $G$ is the matrix in (12), is a generator matrix of an optimal binary $[15,8,(5,5,5,5,4,4,4,4,4,4,4,3)]$ LUEP code.

**Construction S**

The 8 by 15 matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 00000100010000
\end{bmatrix}
$$

is a generator matrix of an optimal binary $[15,8,(5,5,5,4,4,4,4,4,4,4,4,4,4,4,4)]$ LUEP code.

**REFERENCES**