CALCULATION FOR THE DIFFRACTION EFFICIENCY OF ACOUSTO-OPTIC MODULATORS USING AN INTEGRAL EQUATION APPROACH

by A. J. FOX

Philips Research Laboratories, Redhill Surrey RH1 5HA, England

Abstract

The integral equation approach is used to calculate the diffraction efficiency of an acousto-optic modulator. The method reduces the scattering equation to a Fourier transform of the equivalent plane wave in the interaction region. The transform approach simplifies the calculation of diffraction efficiencies in the far field region and an explicit expression can be given for the efficiency in terms of the interaction parameters. The method is also used to calculate the amplitude of the first order scattered beam and the back-scattered beam. Comparisons are made with the results given by plane wave theories and it is shown that good agreement is obtained. The results are also applied to the case of time varying inputs in order to obtain an expression for the transfer function. However, although an integral expression can be obtained, it is not possible to simplify the result to an explicit form.

PACS numbers: 42.20.y and 78.20.Hp.

1. Introduction

The calculation of the diffraction efficiency for acousto-optic modulators operating in the Bragg mode has been given by many authors, see refs 1–6. In general two approaches are possible:

(a) a plane wave analysis valid for infinitely wide optical beams
(b) an integral equation method using a modified Huygen’s principle or Green’s function methods.

The first method can be used in the strong coupling case where the scattered amplitude is a non-linear function of acoustic power. This region is of greatest interest for high-efficiency modulators. However, the treatment is not directly applicable to modulators using strongly focussed light beams such as would occur in wide bandwidth, fast rise-time modulators.

The integral equation method, on the other hand, is best solved in the linear small-signal region. However, the method is not restricted to wide optical beams and can be applied to strongly focussed beams. The integral equation
method is often considered to give a more accurate description than the plane-wave approach because of the ability to take detailed account of the dimensions of the interaction volume. The integral equation approach has been adopted in many papers where it can be seen that the method leads to complicated expressions for the scattered wave amplitude \(1,7\). One reason for this is that the method evaluates the amplitude of the scattered wave in the far-field region and therefore includes diffraction spreading. Other methods, using a quasi-plane wave approach \(8\), can evaluate the amplitude of the scattered wave and hence the diffraction efficiency in the near-field region. It is clear that the diffraction efficiencies obtained by the two methods must agree since no energy is lost by the diffraction spreading. However, it is not apparent from the published work that the two methods do result in the same or even approximately the same expressions for the diffraction efficiency. One of the purposes of this paper is to show that the integral equation expressions can be used to calculate the diffraction efficiency of an acousto-optic modulator operating in the small signal region and that there is close agreement with the expressions obtained using a quasi-plane wave theory \(8\). In addition the paper also shows how the method can be used to calculate the amplitude of the scattered beam and also the amplitude of the back-scattered beam which results when there is appreciable power in the scattered beam.

2. The integral equation approach

It is well known that the passage of light through a medium can be regarded as the excitation and re-radiation of energy from elementary dipole sources at the frequency of excitation. These sources emit spherical waves which ultimately form quasi-plane wave fronts propagating in the forward direction. In the integral equation approach to acousto-optic interaction these dipole sources are induced by the simultaneous passage of a light wave and an acoustic wave through the medium. The frequencies of the radiating dipole sources will be at either the sum or the difference of the optical and acoustic frequencies. These sources give rise to scattered or diffracted light beams propagating at an angle to the main or through beam. In general these beams are weak because of destructive interference. However, for certain conditions and directions the scattered beams add coherently and the amplitude increases progressively throughout the interaction volume. This type of scattering is termed Bragg scattering and for the particular case of monochromatic plane waves the Bragg scattering conditions can be written:

\[
\omega_1 = \omega_o \pm \omega_a \quad \text{and} \quad k_1 = k \pm K,
\]

where \(\omega_1, \omega_o\) and \(\omega_a\) are the angular frequencies and \(k_1, k\) and \(K\) the propaga-
A. J. Fox

tion vectors for the scattered optical, incident optical and acoustic beams, respectively. These conditions are equivalent to the conservation of photon/phonon energies and the conservation of vector wave momentum. It can be shown from these equations that the incident beam direction must be at an angle $\theta_B$ to the acoustic wavefronts where

$$\sin \theta_B = \frac{K}{2k}.$$  \hfill (2.2)

The angle $\theta_B$ is termed the Bragg angle.

In order to calculate the field of the scattered wave at a particular point the vector geometry shown in fig. 1 is assumed. Referring to fig. 1 an elementary dipole of strength $p(r',t)$ is located at $P'(r')$ and the radiated field is observed at the point $P(r)$. The interaction volume is assumed to be the volume denoted by $V'$ with the origin of axes at 0. The dipole emits a spherical wave whose amplitude and phase at $P$ is proportional to $\frac{\exp(-jk_1R)}{R}$ where $R = |r - r'|$ is the distance $PP'$ and $k_1$ the propagation constant of the radiated wave. The electric field $E(r)$ produced by a dipole at distance $R$ can be derived from the Hertz vector $H_e(r)$ where

$$H_e(r,t) = \frac{p(r,t - \frac{R}{v})}{4\pi \varepsilon R},$$  \hfill (2.3)

where $\varepsilon$ is the permittivity of the medium, $v$ the velocity of light and $p(r',t - \frac{R}{v})$ is the polarization vector of the dipole at the retarded time $(t - \frac{R}{v})$. By using the following relationship\textsuperscript{8} for the electric field

$$E(r,t) \, 3 = \nabla \times \nabla \times H_e(r,t),$$  \hfill (2.4)

Fig. 1. Vector formulation for Bragg scattering calculation.

![Fig. 1. Vector formulation for Bragg scattering calculation.](image-url)
it can be shown that the electric field at $P$ in the far field or wave zone region is given by

$$E(r,t) = \frac{\exp(-j k_1 R)}{R} \cdot \frac{k_1 \times [p \times k_1]}{4\pi \varepsilon_0 n^2}, \quad (2.5)$$

where $p(r',t)$ is the dipole polarization at time $t$.

M.K.S. units are used and $n$ and $\varepsilon_0$ are the refractive index of the medium and the permittivity of free space, respectively. The required integral equation can be obtained by summing all the dipole contributions throughout the interaction volume $V'$

$$E(r,t) = \int_{V'} \frac{\exp(-j k_1 R)}{R} \cdot \frac{k_1 \times [p \times k_1]}{4\pi \varepsilon_0 n^2} \cdot dV', \quad (2.6)$$

where $p$ is now taken as the dipole polarization vector per unit volume. This equation is valid for the scattered beams using either of the optical frequencies ($\omega_0 \pm \omega_a$). In the following analysis it will be assumed that the scattered beam has the upconverted frequency ($\omega_0 + \omega_a$).

3. Dipole strength and the coordinate system

To proceed further with eq. (2.6) it is necessary to substitute for the vector polarization at $r'$ i.e. $p(r',t)$ and to choose suitable coordinate axes. For the acoustic and optical waves it is assumed that they can be described, in the complex representation, by the following plane wave expressions

$$n(r',t) = n + \Delta \cdot \exp[j(\omega_a t - K \cdot r')],$$

$$E(r',t) = E(r') \cdot \exp[j(\omega_0 t - k \cdot r')], \quad (3.1)$$

where $k$ and $K$ are the optical and acoustic propagation vectors, $n$ the refractive index and $\Delta$ the incremental change in $n$ due to an acoustic wave which is assumed to be derived from a transducer of length $L$ and height $H$. The above expressions are considered to be valid only within the interaction volume $V'$. For regions beyond the interaction volume $V'$, eq. (3.1) will be modified by diffraction spreading. In the case of the acoustic wave the limitation of length and height of the transducer implies that there is a range of propagation constants for the acoustic beam. For the optical beam a similar modification to the plane wave expression is introduced by the transverse Gaussian distribution. For these cases and others occurring in this paper where a plane wave description is valid only over a limited volume the term 'quasi-plane wave' is used. If, however, it is assumed that the plane wave description holds both in the volume $V'$ and in the far field then the situation is similar to that existing in
geometrical optics where diffraction spreading is ignored. For the purposes of this paper the quasi-plane wave description is assumed to be correct within the volume $V'$ and by using eq. (3.1) (see appendix B for details) the dipole polarization vector density can be shown to be given by

$$p(r', t) = n \varepsilon_0 \Delta |E(r')| \cdot \hat{p} \cdot \exp\{j[(\omega_0 + \omega_d) t - (k + K) \cdot r']\}, \quad (3.2)$$

where $\hat{p}$ denotes the unit dipole vector.

This expression (see appendix B) is only valid when the amplitude of the scattered electric field is much smaller than the incident wave amplitude. Suppressing the time dependence in eq. (3.2) and substituting in eq. (2.6) gives

$$E_1(r) = \frac{1}{4\pi n} \int_{V'} \Delta \cdot |E(r')| \cdot \frac{k_1 \times [\hat{p} \times k_1]}{R} \cdot \exp\{-j[(k + K) \cdot r' + k_1 R]\} \, dV'. \quad (3.3)$$

Further simplification of eq. (3.3) is possible by assuming $r \gg r'$ and using the following approximation,

$$R = |r - r'| = \sqrt{r^2 + r'^2 - 2r \cdot r'} = r \left\{1 + \frac{1}{2} \left[\left(\frac{r'}{r}\right)^2 - \frac{2r \cdot r'}{r^2}\right] - \frac{1}{3} \left[\left(\frac{r'}{r}\right)^2 - \frac{2r \cdot r'}{r^2}\right]^2 + \cdots \right\},$$

putting $r = q$ and assuming that $q$ is constant throughout the volume $V'$ gives the following expression

$$R = q - \frac{r \cdot r'}{q} - \frac{(r \cdot r')^2}{2q^3} + \frac{r'^2}{2q}. \quad (3.4)$$

Retaining only the linear terms gives the far-field approximation

$$R = q - \frac{r \cdot r'}{q}.$$

Substituting this in eq. (3.3) assuming $R = q$ yields the result,

$$E_1(r) = \frac{1}{4\pi n} \cdot \frac{\exp(-jk_1q)}{q} \int_{V'} \Delta \cdot |E(r')| \cdot k_1 \times [\hat{p} \times k_1] \cdot \exp\left\{-j\left[(k + K) \cdot r' - \frac{k_1 r \cdot r'}{q}\right]\right\} \, dV'. \quad (3.5)$$

If plane waves are assumed then the direction of the scattered wave will be given by $k_1 = k + K$ (see eq. (2.1)). However, for the case of finite symmetrical beams this direction will represent the mid-point of the scattered beam. It is assumed that the optical wave vector $k$ and the acoustic wave vec-
tor \( K \) satisfy the Bragg equations, hence the angle between the acoustic wave front and the direction of propagation of the incident optical wave is equal to the Bragg angle \( \theta_B \). The scattered beam is then at an angle \( 2\theta_B \) to the incident beam direction. To evaluate eq. (3.5) under these conditions it is necessary to assume a particular coordinate system. One particularly advantageous choice is to take one of the axes parallel to the direction \( k_1 \) and assume that the axes remain fixed for all points \( P' \). This system of axes is shown in fig. 2 where it can be seen that the \( y \)-axis is parallel to \( k_1 \) and the \( x \)-axis is in the plane containing \( k_1 \) and the acoustic vector \( K \). The points \( P, P' \) have coordinates \((x,y,z)\) and \((x',y',z')\) with respect to these axes and it is assumed that \( P \) is in the far field region where \( \rho \gg r' \). Eq. (3.5) can now be expressed in terms of these axes where it is assumed that the incident optical beam propagates at an angle \( \theta_B \) to the acoustic wave fronts and has a transverse Gaussian distribution as follows,

\[
E(r') = E_0 \cdot \exp \left\{ - \frac{1}{w^2} \left[ (x'\cos 2\theta_B + y'\sin 2\theta_B)^2 + z'^2 \right] \right\},
\]

(3.6)

where \( w \) is the Gaussian waist radius. It should be noted that this expression is an approximation which only holds for the volume \( V' \) (i.e. a quasi-plane wave). A true Gaussian mode expression would be valid in the near and far fields without further modification.

For simplicity it is assumed that the dipole polarization vector \( p \) is perpendicular to \( k_1 \) so that the vector product in eq. (3.5) becomes

\[
k_1 \times [\hat{p} \times k_1] = k_1^2 \hat{p} = n^2 k_0^2 \hat{p},
\]

where \( k_o \) is the propagation constant in free space.

Fig. 2. Axes for incidence at the Bragg angle \( \theta_B \).
It may also be shown that the phase term inside the integral of eq. (3.5) is given by
\[ \phi = k_1 \cdot r' - \frac{k_1}{\ell} (r \cdot r') = -\frac{k_1}{\ell} (x'x + zz'). \]

If however the quadratic terms of eq. (3.4) are retained, then
\[ \phi = -\frac{k_1}{\ell} (x'x + zz') + \frac{k_1}{2\ell} (x'^2 + z'^2). \]

The linear terms give rise to the familiar Fraunhofer diffraction phenomena, whereas the linear and quadratic terms together give Fresnel diffraction. Substituting the preceding expression in eq. (3.5) gives for the electric field at \( P \)
\[ E_1(r) = \frac{\Delta E_0}{4\pi} \cdot \exp(-jk_1\rho) \times \]
\[ \int \exp\left\{ -\frac{1}{w^2} [(x\cos 2\theta_B + y'\sin 2\theta_B)^2 + z'^2] \right\} \cdot \exp\left[ j\frac{k_1}{\ell} (x'x + zz') \right] \cdot dV'. \]

(note that the diffraction term is written as \( \exp(\cdot) \) to distinguish it from the amplitude term). The limits to the volume \( V' \) are set by the transducer length \( L \) and height \( H \). For ease of calculation it is assumed that the acoustic beam propagates in the \( x' \) direction and that the beam width extends along the \( y' \) axis between limits \( \pm L/(2\cos \theta_B) \). The limits on \( z' \) are \( \pm \frac{1}{2}H \) and the limits for \( x' \) can be taken as \( \pm \infty \) to include the full width of the Gaussian beam. This assumption is a reasonable one since acousto-optic modulators are always operated under conditions where there is maximum transmission of the through beam. Eq. (3.7) can now be converted to a Fourier transform integral over \( x', z' \) by carrying out the \( y' \) integration and extending the limits to infinity by using the rectangle function \( \text{Rect}(\cdot) \) where
\[ \text{Rect}(\frac{z'}{H}) = \begin{cases} 1, & |z'| < \frac{H}{2} \\ 0, & |z'| > \frac{H}{2} \end{cases} \]

Eq. (3.7) now becomes
\[ E_1(r) = \frac{\Delta w E_0}{8 \sqrt{\pi} \cdot \sin 2\theta_B} \cdot \exp(-jk_1\rho) \int_{-\infty}^{+\infty} g(x', z') \exp\left[ j\frac{k_1}{\ell} (x'x + zz') \right] \cdot dx'dz', \]
Calculation for the diffraction efficiency of acousto-optic modulators

where

\[ g(x', z') = \text{Rect} \left( \frac{z'}{H} \right) \cdot \exp \left[ - \left( \frac{z'}{w} \right)^2 \right] \times \]

\[ \left[ \text{erf} \left( \frac{x' \cos 2\theta_B + L \sin \theta_B}{w} \right) - \text{erf} \left( \frac{x' \cos 2\theta_B - L \sin \theta_B}{w} \right) \right]. \]

Eq. (3.8) can now be written

\[ E_1(r) = A \cdot \frac{\exp(-j k_1 \varrho)}{\varrho} \cdot G(\hat{x}, \hat{z}), \quad (3.9) \]

where

\[ A = \frac{n k_1^2 \Delta w E_o}{8 \sqrt{\pi} \cdot \sin 2\theta_B} \]

and \( G(\hat{x}, \hat{z}) \) is the Fourier transform of \( g(x', z') \) in terms of the new variables

\[ \hat{x} = \frac{k_1 x}{2\pi \varrho}, \quad \hat{z} = \frac{k_1 z}{2\pi \varrho}, \]

where \( \varrho \) is assumed to be constant.

4. Evaluating the amplitude of the scattered field

The amplitude of the scattered field can now be obtained by performing the integration over the volume \( V' \) in eq. (3.8). We consider two cases:

(a) where the incident optical beam is at the Bragg angle \( \theta_B \) and

(b) where the optical beam is at an angle \( \theta \neq \theta_B \).

(a) Optical beam incident at the Bragg angle \( \theta_B \)

In this case eqs (3.6)–(3.8) are valid. The transform of the error functions in eq. (3.8) can be obtained by using the following result (see appendix A).

\[ \int_{-\infty}^{+\infty} \exp(-j2\pi s x) \cdot [\text{erf}(x + \alpha) - \text{erf}(x - \alpha)] \cdot dx = 4\alpha \cdot \text{sinc}(2\pi \alpha s) \exp[-(\pi s)^2]. \]

From eq. (3.8) the integration over \( x' \) gives

\[ I_{x'} = \int_{-\infty}^{+\infty} \left[ \text{erf} \left( \frac{x' \cos 2\theta_B + L \sin \theta_B}{w} \right) - \text{erf} \left( \frac{x' \cos 2\theta_B - L \sin \theta_B}{w} \right) \right] \times \]

\[ \exp \left( j \frac{k_1 \varrho}{2} x' \right) \cdot dx' \]

\[ = \frac{4L \sin \theta_B}{\cos 2\theta_B} \cdot \text{sinc} \left( \frac{k_1 x L \sin \theta_B}{\varrho \cos 2\theta_B} \right) \cdot \exp \left[ - \left( \frac{k_1 x w}{2\varrho \cos 2\theta_B} \right)^2 \right]. \]
The $z'$ integration may also be performed to give

$$I_{z'} = \int_{-\infty}^{+\infty} \text{Rect} \left( \frac{z'}{H} \right) \cdot \exp \left[ - \left( \frac{z'}{w_1} \right)^2 \right] \cdot \exp \left( j \frac{k_1 z z'}{Q} \right) \cdot dz'$$

$$= \frac{\sqrt{\pi}}{2 w_1} \cdot \exp \left[ - \left( \frac{k_1 w z}{2 Q} \right)^2 \right] \cdot \left[ \text{erf} \left( \frac{H}{2 w_1} - j \frac{k_1 w z}{2 Q} \right) + \text{erf} \left( \frac{H}{2 w_1} + j \frac{k_1 w z}{2 Q} \right) \right].$$

The complete expression for the scattered field becomes

$$E_1(r) = \frac{n k_0^2 L w^2 L E_o}{8 \cos \theta_B \cdot \cos 2\theta_B} \cdot \frac{\exp(-j k_1 \varrho/\varrho)}{Q} \cdot \sin \left( \frac{k_1 \sin \theta_B}{Q \cos \theta_B} \right) \cdot \exp \left\{ - \left( \frac{k_1 w}{2 Q} \right)^2 \left[ z^2 + \left( \frac{x}{\cos 2\theta_B} \right)^2 \right] \right\} \cdot \left[ \text{erf} \left( \frac{H}{2 w_1} + j \frac{k_1 w z}{2 Q} \right) + \text{erf} \left( \frac{H}{2 w_1} - j \frac{k_1 w z}{2 Q} \right) \right]. \quad (4.1)$$

At this stage it is convenient to introduce a unified notation to express this result and others which occur in the following sections. This notation is described in detail in appendix C. In particular, eq. (4.1) can be written, using this notation as follows

$$E_1(r) = \frac{\xi L \cdot k_1 w^2 E_o}{4 \cos \theta_B \cdot \cos 2\theta_B} \cdot \frac{\exp(-j k_1 \varrho/\varrho)}{Q} \cdot \sin \left( 2\pi \alpha' \chi \right) \cdot \exp \left\{ - \pi^2 \left[ \left( \frac{\chi}{\cos 2\theta_B} \right)^2 \right] \right\} \cdot \left[ \text{erf} \left( \frac{H}{2 w_1} + j \pi \zeta \right) + \text{erf} \left( \frac{H}{2 w_1} - j \pi \zeta \right) \right].$$

(4.2)

For far-field approximations where $Q \gg w, z$, the imaginary terms in the error functions can be neglected to yield the expression

$$E_1(r) = \frac{\xi L \cdot k_1 w^2 E_o}{2 \cos \theta_B \cdot \cos 2\theta_B} \cdot \frac{\exp(-j k_1 \varrho/\varrho)}{Q} \cdot \sin \left( 2\pi \alpha' \chi \right) \cdot \exp \left\{ - \pi^2 \left[ \left( \frac{\chi}{\cos 2\theta_B} \right)^2 \right] \right\} \cdot \text{erf} \left( \frac{H}{2 w_1} \right).$$

(4.3)

Expressions similar to this have been given by Gordon\(^1\) and Henderson\(^10\). The amplitude may also be evaluated when both the linear and quadratic phase terms are retained. The result is given in appendix D.
(b) Optical beam NOT incident at the Bragg angle

When $\theta \neq \theta_B$ the vector $k_1 = k + K$ is no longer the propagation vector for the scattered beam since $|k_1| \neq \frac{n \omega}{c}$. In addition the angles of the incident and scattered beams are not equal. For these cases the phase term in eq. (3.5) can be evaluated as follows. The vector $k_1$ is taken to be at an angle $\psi$ to the acoustic wave fronts and the $y$-axis is assumed to be in the $k_1$ direction (see fig. 3). For the vector term in eq. (3.5) i.e. $\left(k + K - \frac{k_1 r}{Q}\right)$ we have the following components,

\[
\begin{align*}
[k + K - \frac{k_1 r}{Q}]_x &= -k \sin(\theta + \psi) + K \cos \psi - \frac{k_1 x}{Q}, \\
[k + K - \frac{k_1 r}{Q}]_y &= k \cos(\theta + \psi) + K \sin \psi - \frac{k_1 y}{Q}, \\
&= k[\cos(\theta + \psi) - 1] + K \sin \psi, \\
[k + K - \frac{k_1 r}{Q}]_z &= -\frac{k_1 z}{Q},
\end{align*}
\]

where from fig. 3

\[
\tan \psi = \left(\frac{K - k \sin \theta}{k \cos \theta}\right).
\]
Putting now

\[ K \cos \psi - k \sin(\theta + \psi) = 2k \cos \psi \left[ \frac{K}{2k} - \frac{\sin(\theta + \psi)}{2 \cos \psi} \right] = 2k u \cos \psi, \]

\[ K \sin \psi + k[\cos(\theta + \psi) - 1] = 2k \sin \psi \left[ \frac{K}{2k} - \frac{\sin^2 \left( \frac{\psi + \theta}{2} \right)}{\sin \psi} \right] = 2k v \sin \psi, \]

where

\[ u = \frac{K}{2k} - \frac{\sin(\theta + \psi)}{2 \cos \psi}, \]

\[ v = \frac{K}{2k} - \frac{\sin^2 \left( \frac{\theta + \psi}{2} \right)}{\sin \psi}, \]

the phase term in eq. (3.5) now becomes

\[ \left[ k + K - \frac{k_1 r}{Q} \right] \cdot \rho = k \left[ x' (2u \cos \psi - \frac{x}{Q}) + y'(2v \sin \psi) - \frac{z'z}{Q} \right]. \]

Substituting in eq. (3.7) gives

\[ E_1(r) = \frac{n k_2^2 \Delta E_0}{4\pi} \cdot \exp(-j k_1 Q) \cdot \frac{1}{Q} \int \exp \left[ - \frac{1}{w^2} \left\{ (x' \cos(\theta + \psi) + y' \sin(\theta + \psi))^2 + z'^2 \right\} \right] \]

\[ \cdot \exp \left\{ - j k \left[ x' (2u \cos \psi - \frac{x}{Q}) + y'(2v \sin \psi) - \frac{z'z}{Q} \right] \right\} \cdot dV'. \]

The integral can now be evaluated in a straightforward but lengthy way to give the expression

\[ E_1(r) = \frac{\xi L \cdot k_1 w^2 E_0}{2 \cos \psi \cdot \cos(\psi + \theta)} \cdot \frac{\exp(-j k_1 Q)}{Q} \cdot \text{sinc} \left( \frac{kaL}{2 \cos \psi} \right) \cdot \exp \left\{ - \left[ \frac{k w (2u \cos \psi - \frac{x}{Q})}{2 \cos(\psi + \theta)} \right]^2 \right\} \cdot \left[ \text{erf}(\overline{H} + j \pi \zeta) + \text{erf}(\overline{H} - j \pi \zeta) \right], \]

where

\[ a = (2u \cos \psi - \frac{x}{Q}) \cdot \tan(\theta + \psi) - 2v \sin \psi. \]
Calculation for the diffraction efficiency of acousto-optic modulators

If $\theta = \theta_B = \psi$ (i.e. incidence at the Bragg angle) eq. (4.6) reduces to eq. (4.2), as it should be.

5. Equivalent plane wave representation

The induced dipole sources emit spherical waves whose envelope is substantially planar for those directions where the waves interfere constructively. These 'envelope waves' are planar only over a limited area perpendicular to the direction of propagation. This equivalent plane wave or rather equivalent quasi-plane wave representation forms a valuable approximation to the form of the electric field within the interaction volume $V'$. To derive this approximation consider the scattered electric field $E_1(r)$ in the wave zone i.e. eq. (3.8).

$$E_1(r) = A \cdot \frac{\exp(-j k_1 q)}{q} \int_{-\infty}^{\infty} g(x', z') \exp\left[j \frac{k_1}{q} (x x' + z z')\right] \cdot dx' dz',$$

where

$$A = \frac{n k_0^2 \Delta w E_0}{8 \sqrt{\pi} \cdot \sin 2\theta_B}.$$

This expression is now compared with the Fraunhofer diffraction integral for an aperture where the optical disturbance over the aperture is $G(x', z')$. The disturbance in the far field is given by

$$U_p = -j \frac{\exp(-j k_1 q)}{\lambda q} \int_{-\infty}^{\infty} G(x', z') \cdot \exp[j 2\pi(\tilde{x} x' + \tilde{z} z')] \cdot dx' \cdot dz',$$

where

$$\tilde{x} = \frac{k_1 x}{2\pi q}, \quad \tilde{z} = \frac{k_1 z}{2\pi q}.$$

If $U_p$ is now identified with the electric field $E_1(r)$, it follows that the electric field in the equivalent aperture is given by

$$E_1(r') = G(x', z') = -\frac{\lambda}{j} A \cdot g(x', z'),$$

hence

$$E_1\left(x', \frac{L}{2}, z'\right) = \frac{j \sqrt{\pi} \cdot k_0 \Delta w E_0}{4 \sin 2\theta_B} \cdot \exp\left[- \left(\frac{z'}{w}\right)^2\right]$$

$$\times \left[ \text{erf} \left(\frac{x' \cos 2\theta_B + L \sin \theta_B}{w}\right) - \text{erf} \left(\frac{x' \cos 2\theta_B - L \sin \theta_B}{w}\right) \right].$$

(5.1)
Also for points such that $|y'| \leq \frac{L}{2}$ the electric field is given by

$$E_1(r') = \frac{i}{2} \sqrt{\pi} \xi \frac{w E_0}{2 \sin 2\theta_B} \exp \left[ -\left( \frac{z'}{w} \right)^2 \right] \times \left[ \text{erf} \left( \frac{x' \cos 2\theta_B + y' \sin 2\theta_B}{w} \right) - \text{erf} \left( \frac{x' \cos 2\theta_B - y' \sin 2\theta_B}{w} \right) \right],$$

where $\xi = \frac{1}{2} k_0 \Delta$.

6. Back scattering contribution

When the amplitude of the first-order beam becomes appreciable, back scattering takes place into the zero-order direction. This process is accompanied by a down conversion of the optical frequency $\omega_1 = (\omega_0 + \omega_a)$ back to $\omega_0$. The amplitude of the 'back scatter' beam can be calculated by the same method as that used for the first-order beam. The propagation vector for the back scattered beam is now $k_2 = k_1 - K = k$, i.e. the same direction as the incident Gaussian beam. The electric field of the first order beam in the volume $V'$ is given by a first approximation by the plane wave representation eq. (5.2). As before the $y''$-axis is chosen along the direction of propagation of

![Fig. 4. Axes for back scatter calculation.](image)

the back scattered beam i.e. $k$ (see fig. 4). The new axes $x'', y'', z''$ are related to the old $x', y', z'$-axes by the relations

$$x' = x'' \cos 2\theta_B - y'' \sin 2\theta_B,$$
$$y' = x'' \sin 2\theta_B + y'' \cos 2\theta_B,$$

(6.1)
Substituting this in eq. (5.2) yields for the first order electric field in the plane-wave representation

\[ E_1(r') = \frac{j \sqrt{\pi} k \Delta w E_0}{4 \sin 2\theta_B} \cdot \exp \left[ - \left( \frac{z''}{w} \right)^2 \right] \]

\[ \times \left[ \text{erf} \left( \frac{x''}{w} \right) - \text{erf} \left( \frac{x'' \cos^2 2\theta_B}{w} - \frac{y'' \sin 4\theta_B}{2w} - \frac{L \sin \theta_B}{w} \right) \right]. \]  

(6.2)

Hence the back scattered electric field in the wave zone is given by

\[ E'_i(r) = \frac{j n k^3 A^2 w E_0}{16 \sqrt{\pi} \cdot \sin 2\theta_B} \cdot \exp(-j k \varrho) \]

\[ \times \int \exp \left[ - \left( \frac{z''}{w} \right)^2 \right] \cdot \left[ \text{erf} \left( \frac{x''}{w} \right) - \text{erf} \left( \frac{x'' \cos^2 2\theta_B}{w} - \frac{y'' \sin 4\theta_B}{2w} - \frac{L \sin \theta_B}{w} \right) \right] \]

\[ \times \text{EXP} \left[ \frac{j k}{\varrho} (x x'' + z z'') \right] \cdot \text{dx}'' \text{dy}'' \text{dz}''. \]  

(6.3)

For a first evaluation of this integral we use the small-angle approximation, i.e.

\[ E'_i(r) = K \cdot \frac{\exp(-j k \varrho)}{\varrho} \]

\[ \frac{1}{\sqrt{\pi} \theta_B} \cdot \int \exp \left[ - \left( \frac{z''}{w} \right)^2 \right] \cdot \left[ \text{erf} \left( \frac{x''}{w} \right) - \text{erf} \left( \frac{x'' - 2y'' \theta_B - L \theta_B}{w} \right) \right] \]

\[ \times \text{EXP} \left[ \frac{j k}{\varrho} (x x'' + z z'') \right] \cdot \text{dx}'' \text{dy}'' \text{dz}''. \]  

(6.4)

where

\[ K = \frac{j n k^3 A^2 w E_0}{32 \sqrt{\pi} \theta_B}. \]

The integration limits are given by \( x'' = \pm \infty, y'' = \pm \frac{1}{2} L, z'' = \pm \frac{1}{2} H. \)

Performing the \( y'' \) integration yields

\[ E'_i(r) = \frac{K}{2\theta_B} \cdot \frac{\exp(-j k \varrho)}{\varrho} \]

\[ \frac{1}{\sqrt{\pi} \theta_B} \cdot \int \exp \left[ - \left( \frac{z''}{w} \right)^2 \right] \cdot \left\{ (2L \theta_B - x'') \left[ \text{erf} \left( \frac{x''}{w} \right) + \text{erf} \left( \frac{2L \theta_B - x''}{w} \right) \right] + \right. \]

\[ \left. \frac{w}{\sqrt{\pi}} \left[ \exp \left[ - \left( \frac{2L \theta_B - x''}{w} \right)^2 \right] - \exp \left[ - \left( \frac{x''}{w} \right)^2 \right] \right] \right\} \times \]

\[ \text{EXP} \left[ \frac{j k}{\varrho} (x x'' + z z'') \right] \cdot \text{dx}'' \text{dz}''. \]  

(6.5)
This is a Fraunhofer diffraction integral where the amplitude of the equivalent plane wave in the volume $V'$ is given by

$$E_i(r'') = -\frac{(k_0 \Delta)^2 w \sqrt{\pi} E_0}{32 \theta_B^2} \cdot \exp\left[-\left(\frac{z''}{w}\right)^2\right]$$

$$\times \left\{ (2L \theta_B - x'') \left[ \text{erf}\left(\frac{x''}{w}\right) + \text{erf}\left(\frac{2L \theta_B - x''}{w}\right) \right] \right\}$$

$$+ \frac{w}{\sqrt{\pi}} \left[ \exp\left(- \left(\frac{2L \theta_B - x''}{w}\right)^2 \right) - \exp\left[-\left(\frac{x''}{w}\right)^2\right] \right].$$

(6.6)

If we now take $x'' = (x + L \theta_B)$ [i.e. change of axes] so that $y''$ lies along the transducer plane and put $\xi = \frac{1}{2} k_0 \Delta$, eq. (6.6) becomes

$$E_i(r'') = -\frac{\xi^2 w \sqrt{\pi} E_0}{8 \theta_B^2} \cdot \exp\left[-\left(\frac{z''}{w}\right)^2\right]$$

$$\times \left\{ (x - L \theta_B) \left[ \text{erf}\left(\frac{x - L \theta_B}{w}\right) - \text{erf}\left(\frac{x + L \theta_B}{w}\right) \right] \right\}$$

$$+ \frac{w}{\sqrt{\pi}} \left[ \exp\left(- \left(\frac{x - L \theta_B}{w}\right)^2 \right) - \exp\left[-\left(\frac{x + L \theta_B}{w}\right)^2\right] \right].$$

(6.7)

This expression agrees with that given by previous plane wave theory.

In order to evaluate the back scatter field in the wave zone we write eq. (6.5) as follows

$$E_i(r) = K \cdot \frac{\exp(-j k q)}{q} \times$$

$$\int_{y''} \exp\left[-\left(\frac{z''}{w}\right)^2\right] \left\{ L \cdot \text{erf}\left(\frac{x''}{w}\right) + \frac{w}{2 \theta_B} \left[ \left(\frac{2L \theta_B - x''}{w}\right)^2 \right] \right\}$$

$$\text{erf}\left(\frac{2L \theta_B - x''}{w}\right) + \frac{1}{\sqrt{\pi}} \cdot \exp\left[-\left(\frac{2L \theta_B - x''}{w}\right)^2\right] - \frac{w}{2 \theta_B}$$

$$\times \left[ \left(\frac{x''}{w}\right) \cdot \text{erf}\left(\frac{x''}{w}\right) + \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{x''}{w}\right)^2\right] \right] \cdot \exp\left[\frac{j k}{q} (xx'' + z z'')\right] \cdot dx'' dz''.$$

(6.8)

This rather formidable integral can be readily evaluated using known transforms (see appendix A). Evaluating the $x''$ integral gives

$$I(x) = \frac{JwL}{\pi X} \left( 1 - \exp[-(\pi \chi)^2] \right) \cdot \left[ 1 - \exp[-j 2\pi \alpha \chi] \cdot \text{sinc}(2\pi \alpha \chi) \right],$$
Calculation for the diffraction efficiency of acousto-optic modulators

where
\[ \alpha = \frac{L \theta_B}{w} \quad \text{and} \quad \chi = \frac{k x w}{2 \pi \varrho}. \]

Evaluating the \( z'' \) integral yields
\[
I(z) = \frac{\sqrt{\pi} w}{2} \cdot \exp[-\pi \zeta^2] \left[ \text{erf}(H + j \pi \zeta) + \text{erf}(H - j \pi \zeta) \right],
\]
where
\[ \zeta = \frac{k z w}{2 \pi \varrho}. \]

Hence
\[
E_1'(r) = -\frac{k E_o (\zeta L w)^2}{8} \cdot \frac{\exp(-j k \varrho)}{\varrho} \cdot (1 - \exp[-(\pi \chi)^2]) \cdot \frac{1 - \exp(-j 2\pi \alpha \chi) \cdot \text{sinc}(2\pi \alpha \chi)}{2\pi \alpha \chi} \cdot \exp[-(\pi \zeta)^2] \cdot \text{erf}(H + j \pi \zeta) + \text{erf}(H - j \pi \zeta).
\]

It can be seen that the \( z \)-distribution is identical to that for the first order scattered field \( E_1(r) \). The \( x \)-distribution, however, departs considerably from the original Gaussian mode. To the distribution given by eq. (6.9) we must add the original Gaussian beam distribution. It can be readily shown that the Gaussian through beam at a distance \( \varrho \) from the origin is given by
\[
E_o(r) = -\frac{j k w^2 E_o}{2} \cdot \frac{\exp(-j k \varrho)}{\varrho} \cdot \exp[-\pi^2(\chi^2 + \zeta^2)].
\]

7. Calculation for the diffraction efficiency

To obtain the diffraction efficiency it is necessary to calculate the total power in the scattered beam. The power flow per unit area for the scattered beam is given by the expression
\[
\frac{dP_1}{dS} = \frac{n}{2} \sqrt{\frac{\varepsilon_o}{\mu_o}} \cdot |E_o|^2,
\]
where \( S \) denotes the cross-sectional area normal to the beam. The total power is then obtained by integration over the surface of a sphere of radius \( \varrho \). Since, however, most of the power is confined to a small range of angles around the direction \( k \), the scattered power can be obtained from the following expression.
\[ P_1 = \int_{-\infty}^{+\infty} \frac{dP_1}{dS} \cdot dx \, dz = \frac{n}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \int |E(r)|^2 \cdot dx \, dz \]

\[ = \frac{n}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \left( \frac{2\pi A}{k_1} \right)^2 \cdot \int \int |G(\tilde{x}, \tilde{z})|^2 \cdot d\tilde{x} \, d\tilde{z}. \quad (7.2) \]

Again, since most of the diffracted power is confined to the direction \( k_1 \), the limits of integration may be extended to \( \pm \infty \) without significant error. By Rayleigh's theorem \(^9\) eq. (7.2) may be related to the function \( g(x', z') \) of eq. (3.8) i.e.

\[ \int_{-\infty}^{+\infty} |g(x', z')|^2 \cdot dx' \, dz' = \int_{-\infty}^{+\infty} |G(\tilde{x}, \tilde{z})|^2 \cdot d\tilde{x} \, d\tilde{z}. \quad (7.3) \]

The power in the through beam is given by

\[ P_o = \frac{n}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \frac{\pi w^2 E_0^2}{2}. \quad (7.4) \]

The diffraction efficiency is therefore given by

\[ \eta = \frac{P_1}{P_o} = \frac{(k_0 A)^2}{8 \sin^2 2\theta_B} \int_{-\infty}^{+\infty} |g(x', z')|^2 \cdot dx' \, dz'. \quad (7.5) \]

The \( z' \) integration may be performed to yield

\[ \int_{-\infty}^{+\infty} \text{Rect} \left( \frac{z'}{H} \right) \cdot \exp \left[ - \left( \frac{\sqrt{2} z'}{w} \right)^2 \right] \cdot dz' = \frac{\sqrt{\pi}}{2} \cdot w \cdot \text{erf} \left( \frac{H}{\sqrt{2} w} \right). \]

Using the unified notation of appendix C gives for the diffraction efficiency

\[ \eta = \sqrt{\frac{\pi}{2}} \cdot \frac{(\xi L)^2 \cdot \text{erf}(\sqrt{2} H)}{8 \alpha^2 \cdot \cos^2 \theta_B \cdot \cos 2\theta_B} \int_{-\infty}^{+\infty} [\text{erf}(p' + \alpha') - \text{erf}(p' - \alpha')]^2 \cdot dp', \quad (7.6) \]

where

\[ p' = \frac{x' \cos 2\theta_B}{w}. \]

It can be shown that the integral reduces to the following

\[ \int_{-\infty}^{+\infty} [\text{erf}(p' + \alpha') - \text{erf}(p' - \alpha')]^2 \cdot dp' \]

\[ = 8 \alpha' \cdot \text{erf}(\sqrt{2} \alpha') + 4 \sqrt{2} \frac{\pi}{\alpha^2} \cdot \exp(-2 \alpha'^2) - 1. \]
The expression for the diffraction efficiency now becomes

\[ \eta = \frac{(\xi L)^2 \cdot \text{erf}(\sqrt{2}H)}{\cos^2 \theta_B \cdot \cos 2 \theta_B} \cdot \left[ \frac{2}{\pi} \cdot \frac{\text{erf}(\sqrt{2} \alpha')}{\alpha'} + \frac{\exp(-2 \alpha'^2)}{2 \alpha'^2} \right]. \tag{7.7} \]

This expression is very similar to the result obtained using the quasi-plane wave theory. If the angle \( \theta_B \) is small so that \( \cos \theta_B, \cos 2 \theta_B \to 1 \) and if \( H \) is large then the expressions become identical. The error function term does not occur in the plane-wave treatment because of the restriction to two dimensions. It can however be easily derived by integrating the power of the Gaussian beam between \( \pm \frac{H}{2} \). Thus it can be seen that the integral equation approach and the plane wave method lead to almost identical expressions for the diffraction efficiency of the first order mode. That this is the case is perhaps not surprising since the integral equation method gives a far field result and it can be seen that eq. (3.8) is equivalent to transforming the electric field in volume \( V' \). The scattered wave can still be regarded as a plane wave in this region, provided that the interaction volume dimensions are of the order of many optical wavelengths.

8. Diffraction efficiency for time varying inputs

It is of some interest to enquire if the method adopted in deriving eq. (7.7) can be extended to obtain the relationship between the modulation on the output beam and the input waveform i.e. the transfer function. This problem is considered here and it is shown that the transfer function can be expressed in an integral equation form which will, in general, require numerical integration.

For the time varying input eq. (3.1) can be modified to include the time variation of the acoustic wave \( \Delta(t) \). This can be related, at least for a 50 Ohm input impedance, to the amplitude of the applied RF voltage \( V(t) \) as follows

\[ \Delta(t) = \frac{\sqrt{M_2}}{2S} \cdot \frac{V(t)}{10}, \tag{8.1} \]

where \( M_2 \) is the acousto-optic figure of merit (see ref. 2) and \( S \) the cross-sectional area of the acoustic beam. Eq. (3.7) for the time varying scattered electric field can be written as

\[
E(r, t) = \frac{n k^2 E_0}{4\pi} \cdot \frac{\exp(-j k_1 \varrho)}{\varrho} \times \int_{V'} \Delta\left[ t - \frac{(x' \cos \theta_B + y' \sin \theta_B)}{u} \right],
\]

\[
\exp\left\{ -\frac{1}{w^2} \left[ (x' \cos 2\theta_B + y' \sin 2\theta_B)^2 + z'^2 \right] \right\} \cdot \text{EXP}\left[ \frac{j k_1}{\varrho} (x x' + z z') \right] \cdot \text{d}V', \tag{8.2}
\]

where \( u \) is the acoustic velocity.
Proceeding as before the $y'$ integration can be carried out to give

$$E(r,t) = \frac{n k_o^2 E_o}{4\pi} \cdot \exp(-j k_1 \phi) \cdot \int_{-\infty}^{+\infty} g(x', z', t) \cdot \exp \left[ \frac{j k_1}{\phi} (x x' + z z') \right] \cdot dx' dz',$$

(8.3)

where

$$g(x', z', t) = \text{Rect} \left( \frac{z'}{H} \right) \cdot \exp \left[ - \left( \frac{z'}{w} \right)^2 \right].$$

As before (see eq. (3.9) the integral in eq. (8.3) can be written as a Fourier transform $G(i, i, t)$ where $\phi$ is assumed to be constant.

From eq. (7.2) the scattered power $P_1(t)$ is given by

$$P_1(t) = \frac{n}{2} \sqrt{\frac{\mu_o}{\mu}} \cdot \left( \frac{k_o E_o}{2} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |G(x, z, t)|^2 \cdot dx \cdot dz.$$

(8.4)

Hence the diffraction efficiency now becomes

$$\eta(t) = \frac{k_o^2}{2\pi w^2} \int_{-\infty}^{+\infty} |g(x', z', t)|^2 \cdot dx' \cdot dz'.\quad(8.5)$$

In general the integral occurring in eq. (8.5) cannot be evaluated in a closed form and even for simple input waveforms it is necessary to resort to numerical integration. As an example the input waveform can be taken to be a simple step function $H(t)$,

$$\Delta(t) = \Delta \cdot H(t) \quad \text{where} \quad H(t) = 1, \quad t > 0$$

$$= 0, \quad t < 0. \quad(8.6)$$

Substituting this in eq. (8.5) and performing the necessary algebra shows that the time variation of the diffraction efficiency is given by

$$\eta(t) = \sqrt{\frac{\pi}{2}} \cdot \frac{(\xi L)^2}{8\alpha^2} \cdot \text{erf}(\sqrt{\frac{H}{\xi}}) \times$$

$$\left[ \int_{-\infty}^{(u t/w + \alpha/2)} \left[ \text{erf}(p + \alpha) - \text{erf}(p - \alpha) \right]^2 \cdot dp + \int_{(u t/w - \alpha/2)}^{(u t/w + \alpha/2)} \left[ \text{erf} \left( \frac{2x t}{w} - p \right) - \text{erf}(p - \alpha) \right]^2 \cdot dp, \right]$$

(8.7)
where $\theta_B$ is assumed to be small and $p = x'/w$. The integrals in eq. (8.7) generally require numerical integration.

9. Conclusion

An integral equation approach is used to derive an expression for the diffraction efficiency of an acousto-optic modulator in the far-field region. It is shown that by correct choice of axes the integral can be reduced to a two dimensional Fourier transform relating the amplitude in the far field to that in the near field. The transform simplifies the calculation of the diffraction efficiency and enables an explicit expression to be given in terms of the modulator parameters. The expressions obtained for the amplitude and diffraction efficiency of the scattered beam are very similar to those derived in an earlier paper $^8$) using a quasi-plane wave theory. The calculation is also extended to an evaluation of the back scattered field and to the case of a time varying input. It is shown that for time varying inputs an integral equation expression can be formulated which will require numerical evaluation for specific input waveforms.

Appendix A

The following integrals and transforms occur in the paper. The notation used for the Fourier transforms is that adopted in Bracewell’s book (ref. 9).

Integrals

a) $\int_{-\infty}^{x} \exp(-t^2) \cdot dt = \frac{\sqrt{\pi}}{2} \cdot \text{erf}(x)$

b) $\int_{-\infty}^{+\infty} \exp[-(aX)^2] \cdot \text{erf}(ax+b) \cdot dx = \sqrt{\pi} \cdot \text{erf} \left[ \frac{ab}{\sqrt{a^2 + b^2}} \right]$

c) $\int_{-\infty}^{+\infty} \exp(-a^2x^2 + 2\beta x) \cdot dx = \frac{\sqrt{\pi}}{2\alpha} \cdot \exp \left( \frac{\beta^2}{\alpha} \right) \cdot \text{erf} \left( \alpha x + \frac{\beta}{\alpha} \right)$

d) $\int_{-\infty}^{+\infty} [\text{erf}(x + \alpha) - \text{erf}(x - \alpha)]^2 \cdot dx = 8\alpha \cdot \text{erf} \sqrt{2\alpha} + 4 \sqrt{\frac{2}{\pi}} [\exp(-2\alpha^2) - 1]$. 

---

Philip, Journal of Research Vol. 41 No. 3 1986
Fourier transforms (F.T.)

e) $\text{F.T.} (\text{erf} \, x) = -\frac{j}{\pi \, s} \cdot \exp[-(\pi \, s)^2]

f) $\text{F.T.} (\text{sgn} \, x) = -\frac{j}{\pi \, s}

g) $\text{F.T.} \left[ x \, \text{erf} \, x + \frac{1}{\sqrt{\pi}} \cdot \exp(-x^2) \mp x \right] = -\frac{1}{4\pi^2 \, s^2} \{ \exp[-(\pi \, s)^2] - 1 \}

h) $\text{F.T.} \left[ \text{erf}(x + \alpha) - \text{erf}(x - \alpha) \right] = 4\alpha \cdot \text{sinc}(2\pi \, \alpha \, s) \cdot \exp[-(\pi \, s)^2].

Appendix B

*Induced polarization vector*²)

Let the small perturbation in permittivity due to the acoustic wave be $\delta(\omega_a)$ where $\omega_a$ is the acoustic frequency. Let the incident optical field be $E(\omega_o)$ and the scattered field $E'$. The total displacement vector is given by

$$D = D(\omega_o) + D' = [\varepsilon + \delta(\omega_a)] \cdot [E(\omega_o) + E'],$$

so that

$$D(\omega_o) = \varepsilon \, E(\omega_o)$$

$$D' = \varepsilon \, E' + \delta(\omega_a) \cdot [E(\omega_o) + E']$$

$$= \varepsilon \, E' + P'.$$

The induced polarization vector is therefore given by

$$P' = \delta(\omega_a) \cdot [E(\omega_o) + E'].$$

If now it is assumed that the scattered field is small i.e. $|E'| \ll |E(\omega_o)|$, then

$$P' = \delta(\omega_a) \cdot E(\omega_o),$$

or in terms of the change in refractive index

$$P' = 2\varepsilon_o \, n \, E(\omega_o) \cdot dn.$$

If we take

$$n = n + \Delta \cdot \cos(\omega_a \, t - K \cdot r) \quad \text{and} \quad E(\omega_o) = E_o \cdot \cos(\omega_o \, t - k \cdot r),$$

we obtain for the polarization vector, as written in the complex representation,

$$p(\omega_o + \omega_a) = \varepsilon_o \, n \Delta \, E_o \cdot \exp[j[(\omega_o + \omega_a) \, t - (k + K) \cdot r]].$$

Appendix C

For convenience the following notation is adopted and used in the body of the paper to express the various end result equations.
Calculation for the diffraction efficiency of acousto-optic modulators

$L$: transducer length
$H$: transducer height
$w$: Gaussian waist radius
$\theta_B$: Bragg angle
$\Delta$: refractive index excursion
$k_o$: propagation constant in free space
$k, k_1, k_2$: propagation constants in medium of index $n$
$q$: distance from interaction volume $V'$ to point of observation.

The following normalized parameters are used:

$$\tilde{H} = \frac{H}{2w}$$

$$\alpha = \frac{L \theta_B}{w}, \quad \alpha' = \frac{L \sin \theta_B}{w}$$

$$\zeta = \frac{1}{2} k_o \Delta$$

$$\chi = \frac{k_1 x w}{2 \pi q}$$

$$\zeta = \frac{k_1 z w}{2 \pi q}$$

$$p = \frac{x'}{w}, \quad p' = \frac{x' \cos 2 \theta_B}{w}$$

Appendix D

Calculation for the amplitude of the scattered beam under Fresnel diffraction conditions

Eq. (4.1) for the amplitude of the scattered field has been derived retaining only the linear terms of eq. (3.4). If, however, both the linear and quadratic terms are retained then the function $g(x', z')$ of eq. (3.8) is replaced by

$$g(x', z') = \text{Rect} \left( \frac{z'}{H} \right) \cdot \exp \left[ - \left( \frac{z'}{w} \right)^2 \right] \cdot \exp \left[ - \left( j \frac{k_1}{2q} (x'^2 + z'^2) \right) \right]$$

$$\times \left[ \text{erf} \left( \frac{x' \cos 2 \theta_B + L \sin \theta_B}{w} \right) - \text{erf} \left( \frac{x' \cos 2 \theta_B - L \sin \theta_B}{w} \right) \right].$$
The integral of eq. (3.8) may now be evaluated to give

\[ \mathbf{E}(r) = K \cdot \frac{\exp(-j k_1 \ell)}{\ell} \cdot I(x) \cdot I(z), \]

where

\[ I(x) = \frac{\sqrt{2\pi}}{\gamma} \cdot \exp(\gamma x)^2 \times \]

\[ \left\{ \text{erf} \left[ \frac{\beta'}{w} (x \cos 2\theta_B + L \sin \theta_B) \right] - \text{erf} \left[ \frac{\beta'}{w} (x \cos 2\theta_B - L \sin \theta_B) \right] \right\}, \]

\[ I(z) = \frac{\sqrt{\pi}}{2} \cdot \frac{\beta}{\gamma} \cdot \exp((\gamma \beta z)^2) \cdot \left\{ \text{erf} \left[ \gamma \left( \frac{H}{2\beta} - \beta z \right) \right] + \text{erf} \left[ \gamma \left( \frac{H}{2\beta} + \beta z \right) \right] \right\}, \]

\[ K = \frac{n k_0^2 A w E_o}{8 \sqrt{\pi} \cdot \sin 2\theta_B} = \frac{k_1 \xi w E_o}{4 \sqrt{\pi} \cdot \sin 2\theta_B}, \]

where

\[ \gamma = \sqrt{\frac{j k_1}{2\ell}}, \quad \beta = \frac{1}{\sqrt{1 + \frac{1}{(\gamma w)^2}}}, \quad \beta' = \frac{1}{\sqrt{1 + \left( \frac{\cos 2\theta_B}{\gamma w} \right)^2}}. \]

REFERENCES

8) A. J. Fox, Optical and Quantum Electronics 14, 189 (1982).