A METHOD OF DESIGNING ROBUST LINEAR PARTIAL RESPONSE EQUALIZERS

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Abstract
As a 'next-best' alternative for adaptive equalizers, whose application may be precluded by e.g. high data rates and power consumption restrictions, this paper develops a method for the design of fixed linear partial-response equalizers that are optimally robust relative to a mean-square performance measure which takes account of system parameter variations. The method is exemplified for a class of digital magnetic recording systems suffering from timing errors, and is in many instances found to establish significant improvements in timing margin at a modest expense to the nominal performance.

Keywords: Digital communication systems, equalization, intersymbol interference, magnetic recording, partial response techniques, robust equalization.

1. Introduction

Besides intersymbol interference and noise, temporal and piece-by-piece variations of the channel characteristics are frequently an important problem in digital transmission and recording equipment. Application of adaptive equalization methods can be regarded as the most natural solution to this problem. Prior to their advent in the second half of the sixties, other means were sought to make data equalizers proof against channel parameter variations. After work by Tufts and Berger, directed towards reduction of the influence of timing errors, Gonsalves and Tufts established the first general result in this field by deriving the full-response fixed linear data equalizer with an optimum performance (in the mean-square sense) averaged over a given probabilistic channel ensemble. Kaye later generalized this work to channels with nonstationary statistics. In spite of their already recognized value, these findings did not cover partial response techniques, where a well-defined part of the arising intersymbol interference (ISI) is not attacked by equalization, but rather dealt with by means of residue arithmetic. In this way, significant
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Performance advantages may accrue relative to full-response equalization, where all ISI is eliminated prior to detection\(^{15}\). Apart from a recurrence to a restricted segment of the theme\(^{16}\), the robust data equalization problem has failed to spur research efforts in later years, partly due to the preferable alternative provided by adaptive techniques in the historically predominant application scenes of data equalization, viz. radio and line transmission\(^2\). Meanwhile, the related area of robust signal equalization, wherein a fully stationary environment is assumed, has continued to bear fruit\(^{17-19}\). As they neglect the cyclostationary nature of the data transmission process, the solutions concerned are of restricted value to the data equalization problem. For example, it is intrinsically impossible to make equalizers proof against timing errors on the basis of a performance measure which does not distinguish between different sampling phases.

In recent years, digital magnetic and optical recording, with their associated data equalization problems, have become increasingly important disciplines\(^{20,21}\). Particularly in the upcoming generations of consumer-grade digital magnetic audio and video recording equipment, areal information densities are slowly increasing up to the point where ISI and noise together demand the limits of the capabilities of traditional equalization methods\(^{20,22,23}\). While the 'nominal' recording channel is relatively well defined\(^{20,24}\), small variations due to e.g. fluctuating tape-to-head contact and timing errors may cause the conventionally applied equalizers to induce considerable and even intolerable degradations of the transmission quality\(^3\).\(^{15}.\)\(^{23}.\)\(^{25}\). Unfortunately, high data rates (in the order of tens or even hundreds of megabits per second) and power consumption restrictions not seldom preclude adaptive techniques from being applied in these systems. For this reason, a renewed interest can be accredited to the topic of robust equalization. In this context, linear partial response equalization merits special consideration in view of its widespread application and also because its sensitivity figures are generally poorer than those of its full response counterpart\(^3\)\(^,\)\(^{15}.\)\(^{23}.\)\(^{25}\). The present paper deals with this subject. In a treatment which extends the one of Gonsalves and Tufts\(^9\), it develops a method to design fixed linear data equalizers which are optimally robust relative to a mean-square performance measure which takes account of both the partial response used and the potentially occurring parameter variations.\(^*\) The adopted performance measure is first optimized in a discrete-time environment, leading to an instructive closed-form frequency-domain solution. The treatment is then extended into the continuous-time domain by deriving a set of equations which uniquely determines the optimally robust continuous-time

\(^*\) To put the method on a solid mathematical basis, an appendix, contributed by A. J. E. M. Janssen, settles the arising existence and uniqueness questions.
equalizer and by outlining an efficient method for its numerical solution. To illustrate its merits, the so-identified design method is finally used to incorporate an optimum resistance to timing errors in linear data equalizers that are employed in a category of digital magnetic recording systems. Comparing the robust equalizers with their conventional counterparts, in many instances significant increments in timing margin are found to be accompanied by a modest performance degradation in the nominal situation, thus underscoring the usefulness of the method.

2. The robustness problem

Figure 1 depicts a discrete-time model of a system suffering from ISI and noise and plagued by channel parameter variations. (Though restricting the ensemble of solutions and therewith entailing some loss of optimality, this model is chosen here because of its simplicity. Later on we will be concerned with the more general continuous-time situation.)

\[
\begin{align*}
    a_k &\rightarrow \text{channel impulse response } f_k^d \\
    n_k &\rightarrow r_k \\
    \text{linear equalizer } r_k &\rightarrow \hat{b}_k \\
    \text{symbol detector } &\rightarrow \hat{b}_k
\end{align*}
\]

Fig. 1. Discrete-time data transmission system employing linear equalization.

A discrete-time data sequence \(a_k \in \{-1, 1\}\) is applied to a discrete-time channel, which transforms \(a_k\) into an output sequence \(r_k\) of the form

\[
r_k = (a \ast f^d)_k + n_k,
\]

where \(\ast\) denotes linear convolution, \(f^d_k\) is the channel impulse response, and \(n_k\) is an additive Gaussian noise sequence. In what follows, we will assume \(a_k\) to be uncorrelated and statistically independent of \(n_k\) (the former assumption can be easily relaxed so as to accommodate the effect of correlation due to e.g. nonlinear transmission codes\(^8\)). The channel impulse response \(f^d_k\) depends upon a parameter \(\Delta\) belonging to an a priori known set \(S\), which accounts for the presence of channel parameter variations. For instance, in the example to be considered later, \(S\) will be the set of all possible sampling phase errors, and \(f^d_k\) will be the equivalent channel impulse response corresponding to a sampling phase error \(\Delta \in S\).
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For the sake of simplicity, we will restrict consideration to linear reception of the received data sequence. In this case a fixed linear equalizer, having coefficients $c_k, -\infty < k < \infty$, performs a linear filtering operation upon $r_k$, transforming it into an output signal (see fig. 1)

$$\tilde{b}_k = (a * f^d * c)_k + (n * c)_k.$$  \hspace{1cm} (2)

Depending upon the dimensioning of the equalizer, $\tilde{b}_k$ can be made to resemble either the originally transmitted data sequence $a_k$ or a simple linear transformation

$$b_k \triangleq (a * g)_k$$ \hspace{1cm} (3)

thereof as well as possible. The latter case, referred to as partial response equalization\(^1\)), involves the use of a partial response $g_k$, whose \(\mathbb{D}\)-transform

$$g(\mathbb{D}) \triangleq \sum_{k=-\infty}^{\infty} g_k \mathbb{D}^k$$ \hspace{1cm} (4)

normally assumes the form

$$g(\mathbb{D}) = (1 + \mathbb{D})^m (1 - \mathbb{D})^n, \hspace{0.5cm} m, n \geq 0.$$ \hspace{1cm} (5)

(The case $m = n = 0$ corresponds to direct estimation of $a_k$, referred to as full response equalization.) For small values of $m$ and $n$, such as usually applied in practical systems, the transformed data sequence $b_k$ has only few (e.g. 3) amplitude levels. The detection of its estimate $\tilde{b}_k$ can therefore be performed by means of a relatively simple symbol-by-symbol detector, and should be followed by an inverse mapping which attempts to reconstruct $a_k$ from its detected transformation $\tilde{b}_k$. Since responses of the form (5) have zeros exactly on the unit circle, a precoding operation must be added at the transmitting end of the system in order for the inverse mapping to be memoryless, thereby preventing error propagation from occurring. As they are not relevant to the forthcoming analysis, neither the precoding operation nor the inverse mapping are shown in fig. 1.

Among the methods available for dimensioning the equalizer, a tractable and frequently applied one minimizes the mean-square error

$$\varepsilon (A) \triangleq \mathbb{E}[ (\tilde{b} - b)^2 ]$$ \hspace{1cm} (6)
between $\hat{b}_k$ and $b_k^2$, where $E$ denotes expectation. The optimum transfer function

$$C(\Omega) \triangleq \sum_{k=-\infty}^{\infty} c_k \exp(-j2\pi \Omega k)$$

(7)

of the equalizer equals$^{2,15}$

$$C(\Omega) = \frac{F^d(\Omega) G(\Omega)}{|F^d(\Omega)|^2 + N(\Omega)},$$

(8)

where the superscript "" denotes complex conjugation, $N(\Omega)$ represents the power spectral density of $n_k$, and $G(\Omega)$ and $F^d(\Omega)$ represent the Fourier transforms of $g_k$ and $f^d_k$, respectively. Using the nominal value for $A$ results in a system that performs optimally in the absence of variations, but whose satisfactory performance is to no extent guaranteed in all other situations.

Rather than designing an equalizer to perform well for a single value of $A$, one might wish to accredit an importance measure to each possible value of $A \in S$, and devise an equalizer which performs as well as possible averaged over $S$ according to the specified distribution. In this way a well-defined degree of robustness would be incorporated in the design. Relative to a mean-square optimality criterion, we will now define this design problem mathematically, and subsequently derive its solution.

3. Optimally robust linear equalization – the discrete-time case

To accredit weight to the distinct possible values of $A$, we define a distribution function $p(A)$ satisfying the usual constraints

$$p(A) \geq 0 \text{ for all } A,$$

(9)

and

$$\int_{A \in S} p(A) \, dA = 1.$$  

(10)

Normally, $p(A)$ will be selected to attain its maximum for the nominal value of $A$, and to be smaller for other values according to their likelihood of occurrence. For a fixed equalizer, we can define an average mean-square error $\bar{\varepsilon}$ as

$$\bar{\varepsilon} \triangleq \int_{A \in S} \varepsilon(A) p(A) \, dA.$$ 

(11)
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By expressing $\epsilon(\Delta)$ in terms of the known system parameters as

$$\epsilon(\Delta) = \int_{-0.5}^{0.5} \left[ |F^d(\Omega) C(\Omega) - G(\Omega)|^2 + N(\Omega) |C(\Omega)|^2 \right] d\Omega$$

(12)

(cf. appendix A of ref. 15), this expression can equivalently be written as

$$\bar{\epsilon} = \int_{-0.5}^{0.5} \int_{\Delta \in S} \left[ |F^d(\Omega) C(\Omega) - G(\Omega)|^2 + N(\Omega) |C(\Omega)|^2 \right] p(\Delta) d\Delta d\Omega,$$

(13)

and we are looking for the fixed transfer function $C(\Omega)$ which optimizes $\bar{\epsilon}$. This problem can be easily solved by adopting a calculus of variations approach. We write $C(\Omega)$ as

$$C(\Omega) \triangleq \hat{C}(\Omega) + \mu V(\Omega),$$

(14)

where $\hat{C}(\Omega)$ and $V(\Omega)$ represent the optimum transfer function and an arbitrary deviation thereof, respectively. To identify $\hat{C}(\Omega)$, we impose the requirement

$$\frac{\partial \bar{\epsilon}}{\partial \mu} |_{\mu=0} = 0 \quad \text{for all } V(\Omega).$$

(15)

Using eq. (14) in eq. (13), we find after the application of some elementary differentiation rules that

$$\frac{\partial \bar{\epsilon}}{\partial \mu} |_{\mu=0} = 2 \text{Re} \int_{-0.5}^{0.5} V^*(\Omega) \left[ \int_{\Delta \in S} \left\{ (|F^d(\Omega)|^2 + N(\Omega)) \hat{C}(\Omega) \right. \right.$$

$$- F^d(\Omega) G(\Omega) \right\} p(\Delta) d\Delta \big] d\Omega.$$

(16)

In order for this expression to equal zero for all $V(\Omega)$ it is required that the quantity within square brackets vanishes for all $\Omega$. Hence we conclude that

$$\hat{C}(\Omega) = \frac{\int_{\Delta \in S} F^d(\Omega) G(\Omega) p(\Delta) d\Delta}{\int_{\Delta \in S} (|F^d(\Omega)|^2 + N(\Omega)) p(\Delta) d\Delta}.$$

(17)

According to eq. (17), the design of an optimally robust equalizer involves the calculation of an 'average matched filter' having the transfer function

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and an average power density spectrum

\[ |F(\Omega)|^2 \triangleq \int_{\Delta \in S} |F^d(\Omega)|^2 p(\Delta) \, d\Delta. \] (19)

In this notation, the optimum equalizer transfer function assumes the simple and intuitively reasonable form (cf. eq. (8))

\[ \hat{C}(\Omega) = \frac{F^*(\Omega) G(\Omega)}{|F(\Omega)|^2 + N(\Omega)}. \] (20)

Using eq. (20) in eq. (13), the corresponding minimum mean-square error \( \bar{e}_{\text{min}} \) is seen to equal

\[ \bar{e}_{\text{min}} = \int_{-0.5}^{0.5} |G(\Omega)|^2 \left[ 1 - \frac{|F(\Omega)|^2}{|F(\Omega)|^2 + N(\Omega)} \right] d\Omega. \] (21)

4. Optimally robust linear equalization - the continuous-time case

Due to the shift-invariance of the discrete-time problem just studied, the transfer function of the optimally robust-data equalizer as given by eq. (20) has an elegant and simple appearance. Not surprisingly, a similar form was also obtained in ref. 17 using a continuous-time, stationary problem setting. As noted before, in data equalization decisions are taken at well-defined, equidistant moments, reflecting the cyclostationary nature of the data transmission process. By extending the treatment given so far into the continuous-time domain, we shall now uncover the consequences of cyclostationarity upon the resulting equalizer transfer function. To this end, we consider the continuous-time model of fig. 2, wherein a data sequence \( a_k \) (whose samples have a temporal spacing \( T \)) is filtered by a continuous-time channel impulse response \( h^d(t) \) and further corrupted by an additive Gaussian noise signal \( n(t) \).
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The so-arising received signal \( r(t) \) is applied to a linear equalizer with impulse response \( w(t) \), whose output is sampled at the moments \( kT \). This yields a sequence of decision variables \( \tilde{b}_k \) of the form

\[
\tilde{b}_k = \sum_{i=-\infty}^{\infty} a_{k-i} (h^L \ast w)(iT) + (n \ast w)(kT),
\]  

(22)

where \( \ast \) again denotes linear convolution, this time in its continuous-time form. As before, we assume \( a_k \) to be uncorrelated, and statistically independent of \( n(t) \). Using these assumptions, the mean-square error

\[
\varepsilon(\Delta) \triangleq E[(\tilde{b} - b)^2]
\]

(23)
between \( \tilde{b}_k \) and \( b_k = (a \ast g)_k \) is found to equal

\[
\varepsilon(\Delta) = \sum_{i=-\infty}^{\infty} (h^L \ast w)^2 (iT) - 2 \sum_{i=-\infty}^{\infty} g_i(h^L \ast w)(iT) + \sum_{i=-\infty}^{\infty} g_i^2 
+ E[(n \ast w)^2(kT)].
\]

(24)

To be able to express \( \varepsilon(\Delta) \) in frequency-domain form, we define the Fourier transform \( W(f) \) of \( w(t) \) as

\[
W(f) \triangleq \int_{-\infty}^{\infty} w(t) \exp \left( \frac{-j2\pi ft}{T} \right) dt,
\]

(25)
analogously define \( H^L(f) \) to be the Fourier transform of \( h^L(t) \), and furthermore denote the power spectral density of \( n(t) \) as \( N(f) \). Chosen here because it simplifies the forthcoming formulas, the adopted Fourier transform definition is somewhat unconventional, involving a normalized and dimensionless frequency variable \( f \). Applying Parseval’s identity and Poisson’s summation...
formula to eq. (24), we can express $e(\Delta)$ in terms of $H^d(f)$, $W(f)$, $N(f)$ and $G(f)$ (as defined before, with $f$ replacing $\Omega$) as

$$ e(\Delta) = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} H^d(f) W^*(f) H^d(f + n) W(f + n) df $$

$$ - \frac{2}{T} \int_{-\infty}^{\infty} H^d*(f) W^*(f) G(f) df + \int_{-0.5}^{0.5} |G(f)|^2 df $$

$$ + \frac{1}{T} \int_{-\infty}^{\infty} N(f) |W(f)|^2 df. \quad (26) $$

The average mean-square error

$$ \bar{e} \triangleq \int_{\Delta \in S} e(\Delta) p(\Delta) d\Delta \quad (27) $$

thus becomes

$$ \bar{e} = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{R}_n(f) W^*(f) W(f + n) df $$

$$ - \frac{2}{T} \int_{-\infty}^{\infty} \bar{H}^*(f) W^*(f) G(f) df + \int_{-0.5}^{0.5} |G(f)|^2 df $$

$$ + \frac{1}{T} \int_{-\infty}^{\infty} N(f) |W(f)|^2 df, \quad (28) $$

where the average transfer function $\bar{H}(f)$ and frequency autocorrelation function $\bar{R}_n(f)$ of the channel are defined respectively as

$$ \bar{H}(f) \triangleq \int_{\Delta \in S} H^d(f) p(\Delta) d\Delta \quad (29) $$

and

$$ \bar{R}_n(f) \triangleq \int_{\Delta \in S} H^d*(f) H^d(f + n) p(\Delta) d\Delta. \quad (30) $$

To find the equalizer transfer function which optimizes $\bar{e}$, we set
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$$W(f) \triangleq \hat{W}(f) + \mu V(f), \quad (31)$$

where $\hat{W}(f)$ and $V(f)$ represent the optimum transfer function and an arbitrary deviation thereof, respectively. The requirement

$$\frac{\partial \mathcal{E}}{\partial \mu} |_{\mu = 0} = 0 \quad \text{for all } V(f) \quad (32)$$

now serves to identify $\hat{W}(f)$. Making use of elementary differentiation rules, it is easily verified that

$$\frac{\partial \mathcal{E}}{\partial \mu} |_{\mu = 0} = \frac{2}{T} \Re \int_{-\infty}^{\infty} V^*(f) \left[ \frac{1}{T} \sum_{n=-\infty}^{\infty} \bar{R}_n(f) \hat{W}(f+n) - \bar{H}^*(f) G(f) + N(f) \hat{W}(f) \right] df \quad (33)$$

A necessary and sufficient condition for this derivative to be zero for all $V(f)$ is that the quantity in square brackets vanishes for all $f$. This means that

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} \bar{R}_n(f) \hat{W}(f+n) + N(f) \hat{W}(f) = \bar{H}^*(f) G(f) \quad \text{for all } f. \quad (34)$$

According to appendix A, this infinite set of equations has a unique solution $\hat{W}(f)$, provided that mild regularity conditions (including positivity and finiteness of $N(f)$) are met. Generalizing upon the conventional nonrobust case\(^8,15\), eq. (34) indicates that the optimum equalizer can be decomposed into a continuum of matched filters (with prewhitening) and cascaded transversal filters. To see this, we define a $\Delta$-dependent transfer function

$$C^\Delta(f) \triangleq G(f) - \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{W}(f+n) H^\Delta(f+n) \quad (35)$$

which is periodic in $f$ with period 1 and hence describes a (symbol interval-spaced) transversal filter. Combining eqs (35), (34), (30) and (29), the ensemble average of the transfer function $H^\Delta(f)/N(f)$ of the matched filter with prewhitening in tandem with $C^\Delta(f)$ is seen to equal

$$\int_{\Delta \in S} \frac{H^\Delta(f)}{N(f)} C^\Delta(f) p(\Delta) d\Delta = \frac{1}{N(f)} [\bar{H}^*(f) G(f) - \frac{1}{T} \sum_{n=-\infty}^{\infty} \bar{R}_n(f) \hat{W}(f+n)] = \hat{W}(f), \quad (36)$$

thus proving the conjecture.
According to eq. (34), $\hat{W}(f)$ is for any frequency $f$ completely determined by samples of $R_n$, $\bar{H}$ and $G$ taken at $f$ and all integer shifts thereof. For this reason, we are led to define two vectors $U(\Omega)$ and $\check{W}(\Omega)$ and a matrix $M(\Omega)$ having components

$$U_i(\Omega) \triangleq \bar{H}^*(\Omega + i) \, G(\Omega + i) \quad \text{for all } i,$$

$$\check{W}_i(\Omega) \triangleq \check{W}(\Omega + i) \quad \text{for all } i,$$

and

$$M_{i,j}(\Omega) \triangleq \frac{1}{T} \bar{R}_{j-i}(\Omega + i) + \delta_{j-i} N(\Omega + i) \quad \text{for all } i, j,$$

respectively. In terms of these entities, (34) can equivalently be written as

$$M(\Omega) \check{W}(\Omega) = U(\Omega) \quad \text{for all } \Omega.$$ (40)

Imposing little more than the condition $0 < N(\Omega) < \infty$, for all $\Omega$, appendix A proves that this matrix equation has a unique solution

$$\check{W}(\Omega) = M^{-1}(\Omega) \, U(\Omega) \quad \text{for all } \Omega,$$ (41)

where "$^{-1}$" indicates matrix inversion. By solving $\check{W}(\Omega)$ for all $\Omega \in [-\frac{1}{2}, \frac{1}{2}]$, $\hat{W}(f)$ is uniquely determined for all $f$ (theorem A.2 of appendix A rigorously asserts the correspondence of $\check{W}(\Omega)$ with $\hat{W}(f)$). When $\bar{H}(f)$ has negligible content outside the interval $[-L, L]$, then only about $2L + 1$ and $(2L + 1)^2$ components of $U(\Omega)$ and $M(\Omega)$ will essentially differ from zero, respectively. As the bandwith restrictions responsible for intersymbol interference automatically cause $L$ to be relatively small, the numerical solution of eq. (40) will generally be quite practicable. A further saving of computation effort can be obtained by using the prior knowledge that $\hat{w}(t)$ must be real-valued, so that $\hat{W}(f) = \hat{W}^*(-f)$ for all $f$. Hence $\hat{W}$ needs to be determined for positive frequencies only, allowing the set of equations (40) to be roughly halved in size. To avoid aliasing distortion in reconstructing $\hat{w}(t)$ from $\check{W}(\Omega)$, an appropriate spacing has to be selected for the frequencies at which $\hat{W}$ (and hence $M$ and $U$) are sampled, related to the temporal extent of $\hat{w}(t)$ as described in ref. 26, pp. 74-76. Mutatis mutandis, a similar remark applies to the reconstruction of convolved versions of $\hat{w}(t)$, such as overall system impulse responses of the form $(h^4 \ast \hat{w})(t)$. 

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Finally, using eqs (34) and (41) in eq. (28), the minimum mean-square error $\bar{E}_{\min}$ achieved by the optimally robust continuous-time equalizer is found to equal

$$\bar{E}_{\min} = \int_{-0.5}^{0.5} |G(f)|^2 \, df - \frac{1}{T} \int_{-\infty}^{\infty} \bar{H}^*(f) G(f) \bar{W}^*(f) \, df$$

$$= \int_{-0.5}^{0.5} \left[ |G(f)|^2 - \frac{1}{T} U(f) M^{-1}(f) U^*(f) \right] \, df,$$

(42)

where 'T' and ',*' indicates transposition and component-wise complex conjugation, respectively. This expression is reminiscent of its discrete-time counterpart eq. (21).

5. A design example: resistance to timing errors in digital magnetic recording

As a comprehensive example, we now apply the developed theory to a class of digital magnetic recording system suffering from timing errors. Conforming to a sizeable fraction of commercially available equipment, we confine ourselves to systems which use an NRZ-like transmission code, longitudinal magnetization and a differentiating playback head \(^{20}\). When both the medium thickness and the gap size of the playback head are sufficiently small, the nominal channel transfer characteristic $H^0(f)$ of a recording system within this category assumes the form \(^{24}\)

$$H^0(f) = [1 - \exp(-j 2\pi f)] \exp(-\pi D|f|) \text{ for all } f,$$

(43)

where $D$ is a normalized measure of the information density on the recording medium, ranging between roughly 0.1 and 3 in current systems. The noise signal $n(t)$ present in the output signal of the playback head normally has a Gaussian distribution and a relatively flat spectral density \(^{23,20}\), taken equal to $N_0$ for the purpose of the present analysis. For the partial response $g_k$, we will consider the popular cases

$$g(\mathcal{D}) = 1,$$

(44)

$$g(\mathcal{D}) = 1 - \mathcal{D},$$

(45)

and

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whose application is referred to as Integrated Detection, Amplitude Detection and Partial Response Class IV Detection, respectively \(^3, 23\).

Having an optimum performance when its output is sampled at the nominal instants \(kT\) (cf. fig. 2), the conventional minimum mean-square error (MMSE) equalizer often incurs significant performance losses in the presence of sampling phase errors, i.e. when sampling erroneously takes place at the instants \((k + \Delta)T, \Delta \neq 0\) \(^{24}\). The influence of sampling phase deviations can be accounted for by incorporating the time shift \(\Delta T\) in eq. (43) as

\[
H^d(f) = H^0(f) \exp \left( j 2\pi \Delta f \right) = \left[ 1 - \exp(-j 2\pi f) \right] \exp(-\pi D|f|) \exp(j 2\pi \Delta f) \quad \text{for all } f. \tag{47}
\]

To delimit the ensemble of possible channel characteristics, we define the set \(S\) to comprise the continuum of all sampling phase errors that are likely to occur in practice, i.e.

\[
S = [-\Theta, \Theta], \tag{48}
\]

where \(0 \leq \Theta \leq \frac{1}{2}\) determines the maximum possible sampling phase deviation. A typical value of \(\Theta\) would be 0.1 \(^{23}\). Furthermore, the phase error distribution \(p(\Delta)\) is defined to be uniform, i.e.

\[
p(\Delta) = \frac{1}{2\Theta} \quad \text{for all } \Delta \in S. \tag{49}
\]

Hence all possible sampling phase deviations are weighed equally strongly.

Apart from the most natural continuous-time solution (to be considered shortly), one can in principle imagine the use of a discrete-time post-equalizer inserted between the sampler output and the detector input in an attempt to lower the effect of timing errors. Using the foregoing discrete-time results, we found that only for small densities \(D\) (up to about \(D = 0.5\), where a timing error essentially induces a gain decrement \(^{24}\)), can some added resistance be achieved for the partial response schemes by increasing the equalizer gain as a precompensation for the expected gain decrement. In all other cases, discrete-time post-equalization turns out to be totally unrewarding, thus confirming the familiar restricted potential of (even adaptive) symbol-spaced transversal equa-
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lizers to withstand timing errors, and implicitly underscoring the practical importance of fractional tap spacing to accomplish insensitivity improvements.

The optimally robust continuous-time equalizer can be determined from the average transfer functions $\bar{H}(f)$ and $\bar{R}_n(f)$ defined in eqs (29) and (30) of sec. 4. Making use of eqs (47) and (49), these functions are seen to equal

$$\bar{H}(f) = [1 - \exp(-j2\pi f)] \sin^2 \frac{2\pi \Theta f}{2\pi \Theta f} \exp(-\pi D|f|) \quad \text{for all } f$$

and

$$\bar{R}_n(f) = 4 \sin^2(\pi f) \frac{\sin 2\pi \Theta n}{2\pi \Theta n} \exp[-\pi D(|f| + |f + n|)] \quad \text{for all } n \text{ and } f$$

in the recording situation under investigation. As a consequence, the vector $U(Q)$ and matrix $M(Q)$ defined in eqs (37) and (40) of sec. 4 have components

$$U_i(Q) = G(Q)[1 - \exp(j2\pi Q)] \sin^2 \frac{2\pi \Theta (Q + i)}{2\pi \Theta (Q + i)} \exp(-\pi D|Q + i|)$$

and

$$M_{i,j}(Q) = \frac{4}{T^2} \sin^2(\pi Q) \frac{\sin 2\pi \Theta (i - j)}{2\pi \Theta (i - j)} \exp[-\pi D(|Q + i| + |Q + j|)]$$

$$+ N_0 \delta(i - j) \quad \text{for all } i \text{ and } j \text{ and for all } |Q| \leq \frac{1}{2}$$

For large $|i|$ and $|j|$, the behaviour of these expressions is dominated by the contained exponential factors, which are smaller than about 1 percent of their maximum value 1 if $D|Q + i|$ and $D(|Q + i| + |Q + j|)$ are larger than about 1.5. Hence, even for a (small) density $D$ of 0.2, the dimensions of $U(Q)$ and $M(Q)$ can be chosen as small as about 17 and $(17 \times 17)$, respectively, while at a (relatively high) density $D$ of 2.5 these figures reduce to a trivial 3 and $(3 \times 3)$. The numerical identification of the optimally robust equalizer is thus a computationally straightforward undertaking. Rather than in mean-square error terms, we will assess the performance of the equalizer in terms of the more meaningful effective signal-to-noise ratio loss.

To this end, we first note that according to eqs (38) and (41), the noise variance $\sigma^2$ of the equalizer output signal can be written as
\[
\sigma^2 = \frac{N_0}{T} \int_{-0.5}^{0.5} \hat{W}^T(\Omega) \hat{W}^*(\Omega) d\Omega = \frac{N_0}{T} \int_{-0.5}^{0.5} U^T(\Omega) M^{-1T}(\Omega) M^{-1*}(\Omega) U^*(\Omega) d\Omega.
\] (54)

The second factor relevant to the performance of the equalizer is the residual intersymbol interference, which can be assessed for any \( \Delta \) by writing the transfer function \( Q(\Omega) \) of the sampled overall system impulse response \( q_k \) in the form

\[
Q(\Omega) = G(\Omega) + E^d(\Omega).
\] (55)

Defining a vector \( H^d(\Omega) \) to have components

\[
H_i^d(\Omega) \triangleq H^d(\Omega + i) \quad \text{for all } i,
\] (56)

and making use of eq. (38), eq. (55) and Poisson's summation formula, the deviation transfer function \( E^d(\Omega) \) can be seen to attain the form

\[
E^d(\Omega) = H^{dT}(\Omega) M^{-1}(\Omega) U(\Omega) - G(\Omega).
\] (57)

The inverse Fourier transform \( e_\Delta \) of \( E^d(\Omega) \) determines the \( (\Delta\text{-dependent}) \) residual intersymbol interference prior to detection. More specifically, it determines the minimum distance \( d/2 \) between any possible value of the non-noise component \( (a \ast (g + e^d))_k \) of the detector input signal and the nearest detector threshold level. In turn, \( d/2 \) directly governs the error probability at high signal-to-noise ratios as \( \approx 2.27 \)

\[
\Pr[E] \propto Q\left(\frac{d}{2\sigma}\right),
\] (58)

where

\[
Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy.
\] (59)

For integrated detection \( (g(\varnothing) = 1) \), \( d/2 \) can be expressed in \( e_\Delta \) as

\[
\frac{d}{2} = 1 + e_0^d - \sum_{k \neq 0} |e_k^d|,
\] (60)
while for amplitude and partial response class IV detection \( g(\theta) = 1 - \theta^l \) with \( l = 1 \) and 2, respectively)

\[
\frac{d}{2} = 1 - \max\{e_i^d - e_0^d, |e_0^d + e_i^d|\} - \sum_{k \neq 0, l} |e_k^d|, \tag{61}
\]

provided that the detector thresholds are located at \( \pm 1 \).\(^3\)

To assess the performance of the equalizer, we normalize the effective signal-to-noise ratio \((d/2\sigma)^2\) of the decision variable on its optimum value

\[
\text{SNR} \triangleq \frac{1}{N_0T} \int_{-0.5}^{0.5} |F^0(\Omega)|^2 d\Omega
\]

\[
= \frac{2}{\pi N_0T} \frac{1}{D(D^2 + 1)}, \tag{62}
\]

which corresponds to matched filter bound operation \(^{15}\). The effective signal-to-noise ratio loss

\[
\mathcal{L} \triangleq \frac{\text{SNR}}{(d/2\sigma)^2}
\]

is therefore bounded by \( \mathcal{L} \geq 1 \), a value of 1 indicating a performance that cannot be improved upon by any receiver.

Proceeding numerically along the lines just sketched, we have determined for various sets of system parameters the optimally robust equalizer according to (41) and the corresponding \( \Delta \)-dependent effective signal-to-noise ratio loss \( \mathcal{L} \). From these results, we shall first describe those related to the region of low information densities. For a density \( D \) of 0.2 and sampling phase errors ranging from \(-25\) to \(+25\) percent of a bit interval, fig. 3 depicts the effective signal-to-noise ratio loss \( \mathcal{L} \) incurred by several optimally robust continuous-time \( 1 - \theta \) equalizers as well as by a conventional raised cosine \( 1 - \theta \) scheme (the latter curve is taken from ref. 24, and has a bearing upon a transition parameter \( \beta \) of 0.5). The raised cosine equalizer, which in ref. 24 was judged to be relatively insensitive to timing errors, lags far behind the robust equalizers for all values of \( \Theta \) within the scale of fig. 3. Combining large tolerance increments with a modest degradation of the nominal performance, the depicted characteristics convincingly demonstrate the merits of the developed design method. To illustrate their spectral properties, fig. 4 depicts the amplitude-frequency characteristics of the robust equalizers of fig. 3. Apart from having a
somewhat increased average transfer magnitude (and hence noise enhancement), the robust equalizers also de-emphasize frequencies onwards of roughly the bit frequency in an apparent attempt to reduce phase ambiguities of the form \( \exp(2\pi \Delta f) \), whose influence is proportional to \( f \).

Also at a (high) density \( D \) of 2.5, the continuous-time partial response equalizer is capable of establishing a robustness improvement, though considerably smaller in magnitude than at \( D = 0.2 \). Figure 5 illustrates this statement for the \( 1 - \mathcal{D}^2 \) equalizer and values of \( \Theta \) ranging from 0 to 0.15. Interestingly, the \( \Theta = 0.15 \) equalizer virtually coincides with the \( \beta = 0.5 \) \( 1 - \mathcal{D}^2 \) raised cosine scheme studied in ref. 24, whose performance characteristics are also included in fig. 5. This similarity suggests that partial response schemes such as applied in current high-density practice are close to optimally insensitive to timing errors, even though their sensitivity may still be unsatisfactory in absolute terms.

A completely different conclusion applies to the full response (\( g(\mathcal{D}) = 1 \)
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Fig. 4. Amplitude-frequency characteristics $A(f)$ of the optimally robust equalizers of fig. 3. a. $\Theta = 0$; b. $\Theta = 0.1$; c. $\Theta = 0.15$. The frequency has been normalized and the symbol rate $1/T$.

Fig. 5. Effective signal-to-noise ratio loss $\mathcal{L}$ versus normalized sampling phase error $\Delta$ for several continuous-time partial response Class IV $(1 - \Theta^2)$ equalizers. Signal-to-noise ratio SNR = 25 dB; Normalized information density $D = 2.5$. a. Minimum mean-square error equalizer $(\Theta = 0)$; b. Optimally robust equalizer $(\Theta = 0.1)$; c. Optimally robust equalizer $(\Theta = 0.15)$; d. Raised cosine equalizer $(\beta = 0.5)$. 
equalizer, whose resistance to timing errors is invariably favourable to the partial response schemes, and moreover increases significantly with increasing $\Theta$, as fig. 6 reveals. For example, at a cost of only 0.3 dB in nominal performance, the $\Theta = 0.1$ equalizer can withstand sampling phase variations as large as 12 percent of a bit interval before its performance decreases by more than 3 dB relative to its nominal value, as opposed to 8 percent for the conventional ($\Theta = 0$) equalizer. The spectral characteristics responsible for this robustness improvement are visualized in fig. 7. Contrary to the low-density $1 - \Theta$ situation just studied, the improved robustness is now apparently brought about by an increased transfer magnitude onwards of roughly the Nyquist frequency, just below which a decrement can be observed.

More impressively than by means of the preceding graphs, the improved resistance to timing errors of the designed robust equalizers is reflected in their eye patterns. The figs 8 and 9 depict eye patterns that were constructed by computer simulation for the equalizers of figs 3 and 6, respectively. Achieving (nearly) the same noise suppression as the conventional MMSE equalizer and an even better timing margin than the raised cosine equalizer, the robust $1 - \Theta$
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Fig. 7. Amplitude-frequency characteristics $A(f)$ of the equalizers of fig. 6. a. $\theta = 0$; b. $\theta = 0.1$; c. $\theta = 0.15$. The frequency has been normalized on the symbol rate $1/T$.

Fig. 8. Eye patterns for the equalizers of fig. 3. Signal-to-noise ratio SNR = 20 dB; Normalized information density $D = 0.2$. a. Minimum mean-square error equalizer ($\theta = 0$); b. Optimally robust equalizer ($\theta = 0.1$); c. Optimally robust equalizer ($\theta = 0.15$); d. Raised cosine equalizer ($\beta = 0.5$).
equalizers of fig. 8 apparently combine the 'best of both worlds'. Less spectacularly but clearly visible, the robust full response equalizers of fig. 9 achieve a larger timing margin at virtually no expense to the noise enhancement.

6. Concluding remarks

The preceding pages have indicated the feasibility and illustrated the usefulness of equipping both partial and full response linear data equalizers with a well-defined degree of robustness. Explicitly using prior knowledge about the ensemble of possible channel characteristics, the presented equalizer design method involves the calculation of two average channel spectra and the inversion of a set of matrices, whose dimensions are proportional to the channel bandwidth and are usually small. The method optimizes the dimensioning of the linear equalizer relative to a performance measure which takes the form of a mean-square error, averaged by means of a predefined weight function over the nominal channel and all possible deviations thereof.
To assess the merits of the design method, we have applied it to a class of digital magnetic recording systems employing an NRZ-like transmission code in conjunction with a differentiating playback head, and suffering from timing errors. For this category, we observed that considerable robustness improvements are frequently achievable at a modest cost in nominal performance. At low information densities, we found the optimally robust Bipolar \((1 - \mathcal{B})\) equalizer to be much less sensitive than its conventional minimum mean square error counterpart, which suffers a 3 dB loss in effective signal-to-noise ratio for sampling phase deviations as small as 8 percent of a bit interval. Sacrificing some 0.9 or 1.5 dB in the nominal situation, the optimally robust equalizers extended this range to a comfortable 15 and 19 percent, respectively. At high densities, only small robustness improvements were observed for the partial response types of equalizer, indicating that currently applied schemes are close to optimally insensitive, even though on an absolute scale their sensitivity may be unsatisfactory. In contrast, the full response linear equalizer was invariably judged to be distinctly superior to the partial response schemes in terms of robustness, while its robust versions improved the timing margin even further.

As a final remark, we note that the one-dimensional development and examples presented in this paper can be straightforwardly extended to systems in which variations of a tractable multitude of parameters occur.

Appendix A. Existence and uniqueness of solutions of the equations (34) and (40)

In this appendix we shall show that, under proper assumptions on \(H\) and \(N\), the eq. (34) has for every \(G \in L^2(\mathbb{R})\) a unique solution \(\hat{W} \in L^2(\mathbb{R})\). Here \(L^2(\mathbb{R})\) denotes the Hilbert space of square integrable functions on \(\mathbb{R}\) with inner product

\[
(W, V)_{\mathbb{R}} \triangleq \int_{-\infty}^{\infty} W(x)V^*(x) \, dx \quad \text{for} \ W, V \in L^2(\mathbb{R}),
\]

(A.1)

and norm

\[
||W||_{\mathbb{R}} \triangleq (W, W)^{1/2}_{\mathbb{R}}.
\]

(A.2)

Furthermore, we shall show that, under proper assumptions on \(H\), \(G\) and \(N\), the eq. (40) has for every \(\Omega \in [-\frac{1}{2}, \frac{1}{2}]\) a unique solution \(\hat{W}(\Omega) \in L^2(\mathbb{Z})\). Here \(L^2(\mathbb{Z})\) denotes the Hilbert space of square summable sequences on \(\mathbb{Z}\) with inner product
\[ (W, V)_Z \triangleq \sum_{n=-\infty}^{\infty} W(n) V^*(n) \text{ for } W, V \in L^p(Z), \quad (A.3) \]

and norm

\[ \|W\|_Z \triangleq (W, W)^{1/2}_Z \quad (A.4) \]

It will also be shown that the procedure outlined in eqs (40) to (44) is, under proper assumptions, a valid one to obtain the solution of the eq. (34). In theorem A.1 and theorem A.2 the main results of this appendix are given.

Before we proceed to the proofs, we develop some notation. We drop all overhead \(^\sim\), we take \( T \) equal to one for convenience (and without loss of generality) and we use \( x \in R \) instead of \( f \) or \( \Omega \) in eq. (34) or eq. (40). The basic operators in eqs (34) and (40) are

\[ (THW)(x) \triangleq H^*(x) \sum_{n=-\infty}^{\infty} H(x + n) W(x + n) \text{ for } x \in R, W \in L^2(R), \quad (A.5) \]

and

\[ UHW \triangleq (W, H^*_Z) H^* \text{ for } W \in L^p(Z), \quad (A.6) \]

with \( H \in L^2(R) \), \( H \in l^2(Z) \). Indeed, when we set

\[ \bar{T}_H W \triangleq \int_{\Delta \in S} T_{H^*} W p(\Delta) d\Delta \text{ for } W \in L^2(R), \quad (A.7) \]

and

\[ \bar{U}_H W \triangleq \int_{\Delta \in S} U_{H^*_\epsilon} W p(\Delta) d\Delta, \quad (A.8) \]

we can write eqs (34) and (40) as

\[ \bar{T}_H W + N W = \bar{H}^* G \text{ for } W \in L^2(R), \quad (A.9) \]

and
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\[ \overline{U}_H W + N_x W = \overline{H}_x G_x \quad \text{for } W \in l^2(\mathbb{Z}). \]  

(A.10)

Here we have written \( H_x^A \triangleq (H^A(x + n))_{n \in \mathbb{Z}}, \) etc., and \( (N W)(x) \triangleq N(x) W(x), \) for \( x \in \mathbb{R}, \) \( (N_x W)(n) \triangleq N(x + n) W(n), \) etc.. Our aim is to find conditions on \( H, \) \( N \) and \( G \) such that eqs (A.9) and (A.10) have unique solutions \( W \in L^2(\mathbb{R}) \) and \( W \in l^2(\mathbb{Z}). \) This is achieved by requiring that the operators \( \overline{T}_H \) and \( U_{H_x} \) are bounded, self-adjoint, positive definite operators of \( L^2(\mathbb{R}) \) and \( l^2(\mathbb{Z}), \) and by getting rid in a decent way of the multiplication operators \( N \) and \( N_x \) in (A.9) and (A.10). The uniqueness and existence results follow then from the familiar fact that an equation \( T f + f = g \) in a Hilbert space \( \mathcal{H} \) has for every \( g \in \mathcal{H} \) a unique solution \( f \in \mathcal{H} \) whenever the linear operator \( T \) is bounded, self-adjoint and positive definite. In ref. 29 one can find a readable account of the theory of linear operators of a Hilbert space.

Proposition A.1

(i) Let \( H \in L^2(\mathbb{R}), \) and assume that

\[ \Psi_H(x) \triangleq \sum_{n=-\infty}^{\infty} |H(x + n)|^2 \quad \text{for } x \in [-\frac{1}{2}, \frac{1}{2}] \]  

is essentially bounded: there is an \( M > 0 \) such that \( \{ x : \Psi_H(x) > M \} \) is a null set, and the minimum of all these \( M \) is the essential supremum, \( \text{ess sup } \Psi_H, \) of \( \Psi_H. \) Then \( T_H \) is a bounded, self-adjoint, positive definite operator of \( L^2(\mathbb{R}). \) Moreover,

\[ \|T_H\|_\mathcal{H} \triangleq \sup_{W \neq 0} \frac{\|T_H W\|_\mathcal{H}}{\|W\|_\mathcal{H}} \leq \text{ess sup } \Psi_H. \]  

(A.12)

(ii) Let \( H \in l^2(\mathbb{Z}). \) Then \( \overline{U}_H \) is a bounded, self-adjoint, positive definite operator of \( l^2(\mathbb{Z}). \) Moreover,

\[ \|\overline{U}_H\|_z \triangleq \sup_{W \neq 0} \frac{\|\overline{U}_H W\|_z}{\|W\|_z} = \|H\|^2_z. \]  

(A.13)

Proof

(i) Let \( W \in L^2(\mathbb{R}). \) We shall show that \( T_H W \in L^2(\mathbb{R}). \) By periodicity of the series in eq. (A.5) we have
\[
\int_{-\infty}^{\infty} |(T_H W)(x)|^2 \, dx = \int_{0}^{1} \sum_{m=-\infty}^{\infty} |H(x + m)|^2 \sum_{n=-\infty}^{\infty} H(x + n) \, W(x + n) \, dx.
\]
(A.14)

By the Cauchy-Schwarz inequality and the definition of \( \Psi_H \) we get

\[
\int_{-\infty}^{\infty} |(T_H W)(x)|^2 \, dx \leq \int_{0}^{1} \Psi_H^2(x) \sum_{n=-\infty}^{\infty} |W(x + n)|^2 \, dx,
\]
(A.15)

and the right-hand side is smaller than or equal to \( \text{ess sup} \, \Psi_H^2 \cdot ||W||^2 \). This shows that \( T_H W \in L^2(\mathbb{R}) \), the boundedness of \( T_H \) and inequality (A.12).

We next show that \( T_H \) is self-adjoint. Let \( W, V \in L^2(\mathbb{R}) \). We shall show that \( (T_H W, V) = (W, T_H V) \). We have

\[
(T_H W, V) = \int_{-\infty}^{\infty} H^*(x) V^*(x) \sum_{n=-\infty}^{\infty} H(x + n) \, W(x + n) \, dx
\]
\[
= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} H(x) \, W(x) \, H^*(x - n) \, V(x - n) \, dx
\]
\[
= \int_{-\infty}^{\infty} W(x) \left[ H^*(x) \sum_{n=-\infty}^{\infty} H(x - n) \, V(x - n) \right]^* \, dx = (W, T_H V).
\]
(A.16)

We finally show that \( T_H \) is positive definite. Let \( W \in L^2(\mathbb{R}) \). We shall show that \( (T_H W, W) \geq 0 \). We have, with \( V(x) \triangleq H(x) \, W(x) \),

\[
(T_H W, W) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} V(x) \, V^*(x + n) \, dx.
\]
(A.17)

If we let

\[
Z(t) \triangleq \int_{-\infty}^{\infty} V(x) \, V^*(x + t) \, dx \quad \text{for} \ t \in \mathbb{R},
\]
(A.18)

then we find by the Poisson summation formula (ref. 26)

\[
(T_H W, W) = \sum_{n=-\infty}^{\infty} Z(n) = \sum_{n=-\infty}^{\infty} z(n),
\]
(A.19)

where
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\[ z(s) \triangleq \int_{-\infty}^{\infty} Z(t) \exp(-j2\pi st) \, dt \quad (A.20) \]

is the Fourier transform of \( Z \). Since \( z(s) = |\nu(s)|^2 \), with \( \nu \) the Fourier transform of \( V \), it follows that \((T_H W, W)_{\mathcal{H}} \geq 0\), and the proof of (i) is complete.

(ii) This statement is completely trivial.

For the measure-theoretic intricacies in the proof of the following proposition one may consult ref. 30.

**Proposition A.2**

(i) Assume that \( H^d(x) \) is measurable as a function of \((x, \Delta) \in \mathcal{H} \times S\), and that

\[ C_H^2 \triangleq \int_{\Delta \in S} \text{ess sup} \, \Psi_H^2 p(\Delta) \, d\Delta < \infty. \quad (A.21) \]

Then \((A.7)\) defines a bounded, self-adjoint, positive definite operator of \( L^2(\mathcal{H}) \). Moreover,

\[ ||\bar{T}_H||_{\mathcal{H}} \leq C_H. \quad (A.22) \]

(ii) Let \( x \in [-\frac{1}{2}, \frac{1}{2}] \), and assume that \( H^d_s \) is measurable as a function of \( \Delta \in S \). Furthermore, assume that

\[ C_L^2(x) \triangleq \int_{\Delta \in S} \Psi_H^2 p(\Delta) \, d\Delta < \infty. \quad (A.23) \]

Then \((A.8)\) defines a bounded, self-adjoint, positive definite operator of \( \ell^2(\mathbb{Z}) \). Moreover,

\[ ||\bar{U}_{H_s}||_{\mathbb{Z}} \leq C_L. \quad (A.24) \]

**Proof**

(i) The quantity \((T_H^d W)(x)\) is measurable as a function of \((x, \Delta) \in \mathcal{H} \times S\) when \( W \in L^2(\mathcal{H}) \). By the Cauchy-Schwarz inequality

\[ \left( \int_{\Delta \in S} |(T_H^d W)(x)|^2 p(\Delta) \, d\Delta \right)^2 \leq \int_{\Delta \in S} |(T_H^d W)(x)|^2 p(\Delta) \, d\Delta \quad \text{for} \ x \in \mathcal{H}. \quad (A.25) \]

By Fubini’s theorem and proposition A.1 (i)
It follows from Fubini's theorem that $T_h^d W \in L^2(\mathbb{R})$ for allmost all $\Delta$, that

$$
\int_{\Delta \in S} (T_h^d W)(x) p(\Delta) d\Delta
$$

is well-defined for almost all $x \in \mathbb{R}$ and belongs to $L^2(\mathbb{R})$ as a function of $x$. Hence $\tilde{T}_h W$ is well-defined and belongs to $L^2(\mathbb{R})$ for $W \in L^2(\mathbb{R})$, and also the inequality (A.22) follows. The remaining facts can be shown to hold noting that for $W, V \in L^2(\mathbb{R})$

$$
(T_h^d W, V)_R = \int_{\Delta \in S} (T_h^d W, V)_R p(\Delta) d\Delta
$$

and using proposition A.1 (i).

(ii) The proof of this is similar to the proof of (i).

**Proposition A.3**

(i) Assume that $H^d(x)$ satisfies the conditions of proposition A.2 (i) and that $V \in L^2(\mathbb{R})$. There is a unique $W \in L^2(\mathbb{R})$ such that $\tilde{T}_h W + W = V$. Moreover, $||W||_R \leq ||V||_R$.

(ii) Let $x \in [-\frac{1}{2}, \frac{1}{2}]$, assume that $H^d(x)$ satisfies the conditions of proposition A.2 (ii) and that $V \in L^2(\mathbb{R})$. There is a unique $W \in L^2(\mathbb{R})$ such that $\tilde{U}_h W + W = V$. Moreover, $||W||_Z \leq ||V||_Z$.

**Proof**

(i) The existence and uniqueness result follows from general Hilbert space theory, see ref. 29. The inequality $||W||_R \leq ||V||_R$ follows from $||W||_R^2 \leq (\tilde{T}_h W, W)_R + (W, W)_R = (V, W)_R$ and the Cauchy-Schwarz inequality.

(ii) The proof of this is similar to the proof of (i).

We shall now reduce the general cases of eqs (A.9) and (A.10) to the special ones of proposition A.3. To this end we introduce the functions

$$
K^d(x) = \frac{H^d(x)}{N(x)} \text{ and } L^d(x) = \frac{H^d(x)}{N(x)} \text{ for } x \in \mathbb{R}
$$

(A.29)
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where it is assumed that \( N(x) \) is measurable and \( N(x) > 0 \) almost everywhere.

**Theorem A.1**

(i) Assume that \( H^d(x) \) and \( N(x) \) are measurable as a function of \( (x, A) \in \mathcal{F} \times \mathcal{S} \), that \( N(x) > 0 \) almost everywhere, that \( M \overset{\Delta}{=} \text{ess sup} N < \infty \) and that \( C_L < \infty \). For any \( G \in L^2(\mathcal{F}) \) there is a unique \( W \in L^2(\mathcal{F}) \) such that eq. (A.9) holds. Moreover,

\[
|| W ||_{\mathcal{F}} \leq (C_L^{1/2} + C_L C_K M^{1/2}) || G ||_{\mathcal{F}}.
\]  

(ii) Let \( x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \), assume that \( H^d \) is measurable as a function of \( A \in \mathcal{S} \), that \( N(x + n) > 0 \), all \( n \), that

\[
M(x) \overset{\Delta}{=} \sup_n N(x + n) < \infty
\]  

and that \( C_L(x) < \infty \). For any \( G_x \in l^2(\mathcal{Z}) \) there is a unique \( W_x \in l^2(\mathcal{Z}) \) such that eq. (A.10) holds. Moreover,

\[
|| W_x ||_{\mathcal{Z}} \leq (C_L^{1/2} (x) + C_L(x) C_K (x) M^*(x)) || G_x ||_{\mathcal{Z}}.
\]  

**Proof**

(i) Consider the equation

\[
\bar{T}_k Z + Z = Y,
\]  

where \( Y \overset{\Delta}{=} N^{-\frac{1}{2}} \bar{H}^* G \). We shall show that the solutions \( Z \) of (A.33) and \( W \) of (A.9) are in one-to-one correspondence by means of the transformation \( Z = N^{1/2} W \). We first note that \( K \) satisfies the conditions of proposition A.2. (i). This is so since \( C_K \leq MC_L \). We next note that \( Y \in L^2(\mathcal{F}) \). We have, more in particular,

\[
|| Y ||_{\mathcal{F}} \leq C_K^{1/2} || G ||_{\mathcal{F}}.
\]  

It follows that for any \( G \in L^2(\mathcal{F}) \) there is exactly one \( Z \in L^2(\mathcal{F}) \) which satisfies eq. (A.33). We shall now show that \( N^{-\frac{1}{2}} Y \in L^2(\mathcal{F}) \). Indeed, it holds that

\[
N^{-\frac{1}{2}} Y \in L^2(\mathcal{F}), || N^{-\frac{1}{2}} Y ||_{\mathcal{F}} \leq C_L^{1/2} || G ||_{\mathcal{F}}.
\]  

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\[
N^{-\frac{1}{2}} \mathcal{T}_k Z = \mathcal{T}_L(N^{\frac{1}{2}} Z) \in L^2(\mathcal{R}), \quad (A.36)
\]

\[
\| N^{-\frac{1}{2}} \mathcal{T}_k Z \|_R \leq C_L M^{\frac{1}{2}} \| Z \|_R \leq C_L M^{\frac{1}{2}} \| \mathcal{Y} \|_R \leq C_L C^{\frac{1}{2}} M^{\frac{1}{2}} \| G \|_R. \quad (A.37)
\]

Here eq. (A.36) follows readily from the definition. Hence \( N^{-\frac{1}{2}} Z \in L^2(\mathcal{R}) \), and

\[
\| N^{-\frac{1}{2}} Z \|_R \leq \| N^{-\frac{1}{2}} \mathcal{Y} \|_R + \| N^{-\frac{1}{2}} \mathcal{T}_k Z \|_R \leq (C_L^2 + C_L C^{\frac{1}{2}} M) \| G \|_R. \quad (A.38)
\]

By multiplying eq. (A.33) from the left by \( N^{\frac{1}{2}} \) and using \( N^{\frac{1}{2}} \mathcal{T}_k Z = \mathcal{T}_h(W^{\frac{1}{2}} Z) \), we see that \( N^{\frac{1}{2}} Z \) is a solution of eq. (A.9). On the other hand, if \( W \) is a solution of eq. (A.9), then \( N^{\frac{1}{2}} W \in L^2(\mathcal{R}) \) is a solution of eq. (A.33). Together with eq. (A.38), this establishes (i).

(ii) The proof of this is similar to the proof of (i).

Our final task consists of tying the solutions \( W \) of (A.9) and the solutions \( W_x \) of eq. (A.10) together.

**Theorem A.2**

Let \( G \in L^2(\mathcal{R}) \) and assume that \( H \) and \( N \) satisfy the conditions of theorem A.1.(i). Let \( W \) be the solution of eq. (A.9). Then \( G_x \in l^2(\mathbb{Z}) \) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \), and \( H_x, N_x \) satisfy the conditions of theorem A.1.(ii) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Moreover, if we denote \( \bar{W}_x \triangleq (W(x + n))_{n \in \mathbb{Z}} \), then \( \bar{W}_x \in l^2(\mathbb{Z}) \), satisfies (A.10) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \) and \( \bar{W}_x = W_x \) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \).

**Proof**

We have

\[
\int_0^1 \| G_x \|_Z^2 \, dx = \int_0^1 \sum_{n=-\infty}^\infty |G(x + n)|^2 \, dx = \int_{-\infty}^\infty |G(x)|^2 \, dx < \infty. \quad (A.39)
\]

Hence \( \| G_x \|_Z^2 < \infty \) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Moreover, measurability of \( H^Q(x) \) as a function of \((x, \Delta) \in \mathcal{R} \times S \) implies measurability of \( H^Q(x + n) \) as a function of \( \Delta \in S \) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \) and \( n \in \mathbb{Z} \), and

\[
\text{ess sup} \, C^2_L(x) = \text{ess sup} \, \int \Psi^2_L(x) p(\Delta) \, d\Delta \leq \int \text{ess sup} \, \Psi^2_L(x) p(\Delta) \, d\Delta = C^2_L. \quad (A.40)
\]
Similarly,

\[ \sup_{n \in \mathbb{Z}} N(x + n) \leq \operatorname{ess} \sup_{n \in \mathbb{Z}} N \]  \hspace{1cm} (A.41)

for almost all \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Also, \( \tilde{W}_x \in L^2(\mathbb{R}) \) for almost every \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Finally, as the proof of proposition A.2.(i) shows,

\[ (\bar{T}_H W)(x) = \bar{U}_H, \tilde{W}_x \]  \hspace{1cm} (A.42)

for almost all \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Now the theorem follows from the uniqueness part of theorem A.1.(ii).

Note

Theorem A.1 and theorem A.2 can be extended to cover the case where \( N(x) = 0 \) for \( x \) in a set of positive measure. In the assertion of theorem A.1.(i) we get the existence of a \( W \in L^2(\mathbb{R}) \) such that eq. (A.9) holds, and this \( W \) can be made unique by the requirement that \( W(x) = 0 \) whenever \( N(x) = 0 \). In this definition (A.29) and everywhere in the proof of theorem A.1.(i) where \( N^{-\frac{1}{2}} \) is applied, we use the convention that \( 0/0 = 0, a/0 = \infty \) when \( a \neq 0 \). The condition \( C_L < \infty \) implies that, for almost all \( x \in \mathbb{R} \), \( H^d(x) = 0 \) for almost all \( \Delta \in S \) when \( N(x) = 0 \). With this convention the proof of theorem A.1.(i) can be easily adapted. We see e.g. that the solution \( Z \) of the equation \( \bar{T}_K Z + Z = Y \) satisfies \( Z(x) = 0 \) whenever \( N(x) = 0 \), and that eqs (A.35) to (A.37) are still valid. In theorem A.1.(ii) the solution \( \tilde{W}_x \) becomes unique when we require that \( \tilde{W}_x(n) = 0 \) whenever \( N(x + n) = 0 \). In theorem A.2 we must take the unique solutions \( W \) and \( \tilde{W}_x \) of the adapted theorem A.1.

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