INSTANTANEOUS SPECTRAL ANALYSIS FOR A QUANTITATIVE MEASUREMENT OF ULTRASONIC ATTENUATION IN BIOLOGICAL TISSUE*

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Abstract

The aim of quantitative echography is to provide numerical estimates of biological tissue parameters, such as attenuation, scattering strength, sound velocity, which can be clinically significant. In this domain, it has become clear that a careful modelling of the echogram formation process is necessary because, as the probing ultrasonic pulse propagates and interacts with biological structures, it undergoes diffraction effects as well. Hence the meaningful information about the propagation medium is intricately altered by diffraction phenomena induced by finite size transducers. By using a time-frequency formalism, we show that a non-stationnary 'diffraction filter' depending only on the imaging transducer can be defined which allows diffraction unbiasing. Theoretical, numerical and experimental evidence is presented for the need for diffraction correction and for the validity of the proposed solution.

Keywords: acoustic wave diffraction, bioacoustics, echography, frequency-time distribution, Rihaczek distribution, Wigner distribution.

1. Introduction

Conventional echographic systems are mainly designed to yield images of the human body: they provide a geometric description of the organs by virtue of discontinuities in the propagation medium which reveal the contours of the organs. Quantitative echography, on the other hand, is concerned with homogeneous parts of the organs, the parts 'between the discontinuities'. Its
object is to provide measurements of any physical parameter which meaningfully characterizes the tissues, the measurements being performed for small domains where the given parameter can be assumed constant. The value of such a quantitative description is now established and many tissue characterization projects are devoted to the development and validation of methods for quantitative estimation.

One parameter has received considerable attention: the acoustic attenuation in the medium. Several methods have been proposed for its estimation by taking advantage of the non-stationarity of the return signal. The basic idea is the following. As later echoes come from deeper layers of tissue they experience a longer propagation path, and as the propagation effect on an echo depends on the attenuation in the medium, the temporal evolution of the ultrasound signal response contains information about that attenuation.

However, there is another origin for the non-stationarity: the diffraction phenomenon also induces time-dependent spectral changes because it is a space-dependent effect like the attenuation. Since diffraction and attenuation both contribute to non-stationarity, neglecting the former leads to biased estimations of the latter. This paper is devoted to a precise modelling and analysis of these phenomena. To build up a sound theoretical basis for tissue characterization, the first step was to set up a complete three-dimensional model for return signals from soft tissues, taking into account the non-stationarity and the stochastic nature of tissues. The second step was the choice of a powerful method for the analysis of non-stationary signals: we chose the time-frequency energy distribution introduced by Rihaczek. Thanks to its properties, and basing ourselves upon our model, we have derived the mean distribution of energy in both time and frequency, which is the crucial quantity in the approach to tissue characterization described above. It reveals that in the time-frequency domain the diffraction effect can be separated from other effects so that a mean diffraction filter can be theoretically defined and studied independently, as it depends only on the transducer. We have investigated some of its properties and also measured it experimentally and compared the result with its numerical value. We have briefly explained how the usual attenuation estimation algorithms could be modified to remove diffraction bias. This is illustrated by a complete experiment with tissue phantom signals processed with the so-called spectral shift algorithm, for which we have observed a strong diffraction effect introducing biases up to 100%. We show how prior correction with the diffraction filter removes these errors, demonstrating the necessity of diffraction correction.
2. The ultrasound signal — a model

This chapter presents a model of a medium probed by an ultrasound signal. We will describe the different physical phenomena which come into play and propose a mathematical model. Relevant phenomena are dealt with in secs 2 to 8 and the statistical hypotheses in sec 9. Finally in sec 10 the notion of analytic signal is introduced which greatly simplifies the understanding of several questions.

2.1. General framework

The propagation medium can be considered as a continuous, non-dispersive medium defined by a propagation speed \( c \), and an acoustic impedance \( Z_m \), and including a distribution of pinpoint scattering targets. A statistical point of view is fundamental: the position and reflectivity of scatterers are randomly distributed.

The medium is linear in two respects. First, a signal due to an isolated target is considered as the result of a succession of physical phenomena (transduction, propagation, scattering) each with a linear description by a transfer function. The global transfer function is therefore the product of transfer functions associated with each phenomenon. Secondly, multiple scattering is not taken into account. The ultrasound signal must therefore be considered to be a pure superimposition of target signals, each one contributing to the medium’s response independently of the other targets (Born approximation). The models for the transfer functions associated with each phenomenon will now be described.

2.2. Transduction

The generator supplies an electrical pulse to the transducer denoted \( g(t) \), causing a vibration of its forward surface. A piston-mode vibration in which all points of the front surface experience the same normal movement defined by the normal speed law \( v_n(t) \) is assumed. The electro-acoustic transfer function \( I^E(f) \) links \( G(f) \) to \( V_n(f) \) by

\[
V_n(f) = I^E(f) \cdot G(f) .
\]  

This first equation illustrates a notational convention used throughout this text: when a function of time is denoted by a lower case letter its Fourier Transform is implicitly denoted by the corresponding upper case.

Although the same transducer listens to the medium’s response, the electric loading conditions can be changed in the reception mode. As we are not
interested in detailing the transduction process, it is simply specified by another transfer function: if the total force in the spectral domain developed by an incident wave on the forward face of the transducer, denoted $F(f)$, results in the electrical signal $e(t)$, and if the electroacoustic transfer function is labelled $I^R(f)$ then:

$$E(f) = F(f) \cdot I^R(f). \quad (2.2)$$

2.3. Diffraction in non-attenuating media

The acoustic wave is propagated in a medium temporarily considered to be non-scattering and non-attenuating. Assuming a baffled transducer, classical diffraction theory gives for any observation point $r_0$ the acoustic potential $\phi(r_0, t)$ due to transducer vibration. In harmonic conditions the transducer being excited at frequency $f$, the choice of Green's function $|r|^{-1} \exp(2i \pi f |r|/c)$ provides the acoustic potential observed at point $r_0$ by the following surface integral:

$$\Phi(r_0, f) = \int \int_T V_n(r, f) \cdot \frac{\exp(2i \pi f |r_0-r|/c)}{|r_0-r|} \, dr, \quad (2.3)$$

where the integration area $T$ is the transducer’s emitting surface and $c$ is the propagation velocity in the medium. Under the hypothesis of a piston mode, $V_n(r, f)$ reduces to $V_n(f)$ and therefore:

$$\Phi(r_0, f) = V_n(f) \cdot H_0(r_0, f), \quad (2.4)$$

where

$$H_0(r_0, f) = \int \int_T \frac{\exp(2i \pi f |r_0-r|/c)}{|r_0-r|} \, dr. \quad (2.5)$$

So, $H_0(r_0, f)$ is the diffraction transfer function which is also the propagation one, the medium being non-attenuating. From the diffraction transfer function, the diffraction impulse response $h_0(r_0, t)$ is obtained. This is the potential observed at $r_0$ caused by the application of a $\delta$-shaped velocity impulse on the front face after propagation in a non-attenuating medium.

In several cases (circular, plane or focused transducer, plane rectangular transducer) an analytic expression for $h_0(r_0, t)$ exists. For instance, for a simple plane transducer, where $R$ is the transducer radius, $z$ the projection
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of \( r_0 \) on the transducer axis, and \( \rho \) the projection of \( r_0 \) on the plane normal to the axis, it is found \(^1\) that the acoustic potential varies between values 0 and \( c \), according to a characteristic function denoted here as \( \gamma(t) \):

\[
\gamma(t) = \pi^{-1} \cos^{-1} \left( \frac{c^2 \rho^2 - z^2 + \rho^2 - R^2}{2 \rho \sqrt{c^2 \rho^2 - z^2}} \right)
\]

(2.6)

and according to three characteristic times defined as

\[
ct_1 = z, \quad ct_2 = \sqrt{z^2 + (\rho - R)^2}, \quad ct_3 = \sqrt{z^2 + (\rho + R)^2}.
\]

(2.7)

It is found that:

- if \( r_0 \) is 'in the shadow of the transducer' \( (\rho < R) \):

\[
h_0(r_0, t) = \begin{cases} 
0 & -\infty < t < t_1 \\
c & t_1 < t < t_2 \\
c \cdot \gamma(t) & t_2 < t < t_3 \\
0 & t_3 < t < +\infty
\end{cases}
\]

(2.8)

- if \( r_0 \) is 'out of shadow' \( (\rho > R) \) we have:

\[
h_0(r_0, t) = \begin{cases} 
0 & -\infty < t < t_2 \\
c \cdot \gamma(t) & t_2 < t < t_3 \\
0 & t_3 < t < +\infty
\end{cases}
\]

(2.9)

Figs 1 and 2 show the changes in \( h_0(r_0, t) \) when \( r_0 \) is increased in magnitude while keeping a constant angle with the transducer axis. In these two figures \( \frac{1}{c} h_0(r_0, t) \) is plotted for six different values of \( |r_0| \) from 25 mm to 150 mm with a 25 mm step. The transducer radius is set to 5 mm.

In fig. 1 the angle between \( r_0 \) and the transducer axis is kept small (0.01 rad.) so that even for \( |r_0| = 150 \text{mm} \), \( r_0 \) is still located in the shadow of the transducer. As a consequence all the responses begin with a discontinuity and keep their maximum value \( c \) over a certain time interval. The important point to be noticed is that the diffraction impulse responses become shorter and shorter as the observation distance increases. Hence in this case the diffraction effect is to increase the high-frequency part of the signal spectrum versus time.

In fig. 2 the constant angle between \( r_0 \) and the axis is now set to 0.15 rad so that only the first observation point is located in the transducer shadow.
Fig. 1. Diffraction impulse response for 6 different ranges. Observation angle: 0.01 rad. Horizontal axis: time in μs.

Fig. 2. Diffraction impulse response for 6 different ranges. Observation angle: 0.15 rad. Horizontal axis: time in μs.
and exhibits a discontinuity at its beginning. As $r_0$ increases, the response duration remains almost constant and is much longer than in the corresponding plots of fig. 1. This can be interpreted in terms of the more familiar harmonic diffraction theory: the waves with low frequencies cannot be easily focused. This is why low-frequency energy is present more outside the transducer shadow: the diffraction impulse response being smoother in this region, it corresponds to a relatively large low-frequency content. As a result the diffraction effect has an influence on the evolution in time of the spectral content of the backward signal. This phenomenon is of great importance in quantitative echography and its statistical effect is studied in detail in the following.

2.4. Diffraction in attenuating media

Attenuation of ultrasonic waves depends strongly on frequency $f$ and is characterized by the attenuation factor $a(f)$, so that the signal amplitude after a path of length $\Delta x$ decreases by the factor $\exp (-a(f) \Delta x)$. Attenuation is taken into account in the propagation transfer function by introducing into eq. (2.3) an attenuated Green function: $\exp (-a(f) |r| \cdot |r|^{-1} \exp (2i \pi f |r|/c)$ and eq. (2.5) now reads

$$H_{\text{diff}}(r_0, f) = \int \int \frac{\exp (2i \pi f |r_0-r|/c)}{|r_0-r|} \cdot \exp (-a(f) |r_0-r|) \, dr. \quad (2.10)$$

In this new propagation transfer function, both attenuation and diffraction are mixed. However, it is important to be able to separate the diffraction and attenuation effects. This is possible because in the previous integral the complex exponent varies much more rapidly with $r$ than the real exponent. Considering the latter as a constant by giving it an average value $\exp (-a(f) |r_0|)$, the transfer function now reads:

$$H_{\text{att}}(r_0, f) = \exp (-a(f) |r_0|) \int \int \frac{\exp (2i \pi f |r_0-r|/c)}{|r_0-r|} \, dr. \quad (2.11)$$

By virtue of this classical approximation, the propagation transfer function is separable into the product of two position-dependent transfer functions: the one describing the attenuation phenomena and the other merely describing the diffraction in a non-attenuating medium. In the following we shall consistently take:

$$H(r_0, f) = \exp (-a(f) |r_0|) \cdot H_0(r_0, f). \quad (2.12)$$
2.5. Scattering

A scattering model must be based on the nature of the scatterers. This can vary from one type of tissue to another or even within a given type of tissue. Scattering may be due to discontinuities or to the presence of biological structures of various types and sizes. The present state of research in this area has not led to a clear identification of the biological origins of scattering, but the classical theory of scattering suggests that in the range of medical ultrasound frequencies we can assume the scattering transfer function to be proportional to $f^n$ where $n$ can have values between 1 and 2.

However, within the statistical framework used (see 2.8) an accurate determination of the actual scattering transfer function is not necessary, as will appear later. Nevertheless a definition is needed. When $u(t)$ is the scattering pulse signal and $U(f)$ its Fourier transform for a given scatterer, the scattering transfer function $U(f)$ is defined as follows. A localized scatterer in $r_0$ is subject to an incident pressure field whose spectrum is $P_{\text{inc}}(r_0, f)$. Considering the scattered wave as a spherical wave emitted from $r_0$, (this is true if scattering is only due to an inhomogeneity of compressibility, and approximately true in backscattering) and defined by its pressure field spectrum $P_{\text{diff}}(r, f)$, $U(f)$ simply links $P_{\text{diff}}$ to $P_{\text{inc}}$ by

$$P_{\text{diff}}(r, f) = U(f) \cdot \frac{\exp(2i \pi f |r_0-r|/c)}{|r_0-r|} \cdot P_{\text{inc}}(r_0, f). \quad (2.13)$$

In the standard theory of scattering, the point size of the scatterers yields an infinitely short response $u(t)$. As acoustic potential and pressure are simply related through $p = \rho_0 \frac{\partial \Phi}{\partial t}$, we have in the frequency domain:

$$P_{\text{inc}}(r_0, f) = 2i \pi \rho_0 f \Phi(r_0, f), \quad (2.14)$$

where $\rho_0$ denotes the medium's density.

2.6. Inverse propagation

The inverse propagation transfer function can be deduced from the one associated with direct propagation. To shorten the discussion, only the reciprocity theorem and reverse transduction are mentioned here.

Assuming a baffled transducer operating in the piston mode, it is found that if the potential $\Phi(r_0, f)$ created at $r_0$ is linked to the normal speed of
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the front face by \( \Phi(r_0, f) = V_n(f) \cdot H_0(r_0, f) \), then a spherical wave emitted from point \( r_0 \), defined by its pressure field \( P(r, f) = \Pi_0 \cdot \frac{\exp(2i\pi f|r_0-r|/c)}{|r_0-r|} \)
develops on the transducer surface a total force \( F(f) \) given by:

\[
F(f) = 4\pi \cdot \Pi_0 \cdot H_0(r_0, f) \cdot \frac{Z_m}{Z_m + Z_r}
\]

(2.15)

where \( Z_r \) is the transducer radiation impedance at the frequency \( f \) and \( Z_m \) is the impedance of the propagation medium.

2.7. Review of phenomena

We have completed the inventory of all the physical effects which contribute to shaping the ultrasound signal. It remains to link the voltage applied to the transducer upon emission to the voltage registered at its terminals when it operates as a receiver. This is the purpose of this section in which all the space-invariant transfer functions that are of lesser interest are grouped in a common factor so as to isolate the effects which depend on the position and nature of the considered scatterer. The following table gives for

<table>
<thead>
<tr>
<th>physical phenomenon</th>
<th>measured by</th>
<th>denoted in time as</th>
<th>Transfer function and related spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>electrical excitation</td>
<td>voltage at terminals</td>
<td>( g(t) )</td>
<td>( G(f) )</td>
</tr>
<tr>
<td>direct transduction</td>
<td>normal velocity on front face</td>
<td>( v_n(t) )</td>
<td>( V_n(f) = G(f) \cdot e^{jE(f)} )</td>
</tr>
<tr>
<td>potential propagation</td>
<td>acoustic potential at ( r_0 )</td>
<td>( \phi(r_0, t) )</td>
<td>( \Phi(r_0, f) = V_n(f) \cdot H(r_0, f) )</td>
</tr>
<tr>
<td>pressure propagation</td>
<td>pressure at ( r_0 )</td>
<td>( P_{inc}(r_0, f) )</td>
<td>( P_{inc}(r_0, f) = 2i\pi \rho_0 \cdot \Phi(r_0, f) )</td>
</tr>
<tr>
<td>scattering</td>
<td>pressure of the scattered wave</td>
<td>( P_{diff}(r_0, t) )</td>
<td>( P_{diff}(r_0, f) = P_{inc}(r_0, f) \cdot U(f) )</td>
</tr>
<tr>
<td>backward propagation</td>
<td>total force on the transducer face</td>
<td>( f(r_0, t) )</td>
<td>( F(r_0, f) = P_{diff}(r_0, f) \cdot H(r_0, f) \cdot \frac{4\pi Z_m}{Z_m + Z_r} )</td>
</tr>
<tr>
<td>inverse transduction</td>
<td>electric voltage captured at terminals</td>
<td>( e(r_0, t) )</td>
<td>( E(r_0, f) = F(r_0, f) \cdot |r_0 \cdot f) )</td>
</tr>
</tbody>
</table>
each physical phenomenon the notation used, the time function describing it, and the associated transfer function. Their product yields

\[ E(r_0, f) = G(f) I^E(f) H(r_0, f) 2i \pi \rho_0 f U(f) H(r_0, f) \frac{Z_m}{Z_m + Z_r} 4\pi I^R(f) \]

(2.16)

which, by setting

\[ M(f) = G(f) I^E(f) I^R(f) \frac{4\pi Z_m}{Z_m + Z_r}, \]

(2.17)

reduces to

\[ E(r_0, f) = M(f) U(f) H^2(r_0, f). \]

(2.18)

The physical meaning of each of these three factors is the following:

- \( M(f) \) is a constant of the considered system: electronic emission-reception, transducer, propagation medium (defined by \( \rho_0, Z, c \));
- \( H(r_0, f) \) depends only on the scatterer's position, for a given geometry, and on two characteristics of the medium: attenuation \( a(f) \) and velocity \( c \);
- \( U(f) \) depends on the nature of the scatterer.

\( H \) can be factorized further, thereby giving

\[ E(r_0, f) = M(f) U(f) \exp(-2a(f)|r_0|) H_0^2(r_0, f) \]

(2.19)

where \( H_0 \) depends only on the transducer's geometry and \( c \) and the attenuation term depends only on the medium.

2.8. Statistical model

Two kinds of parameters are to be considered as random: the positions of the scatterers and their reflectivities. The model is based on simple but reasonable assumptions. The examined volume contains a distribution of pinpoint scatterers of volume density \( n \) in which the position of each scatterer is independent of positions of the other scatterers. Hence probability density for each scatterer is uniform throughout the volume. Secondly, the scattering properties are assumed to be statistically independent of the position. In other words, scattering is not correlated with position.
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In sec. 2.5 the scattering pulse signal \( u(t) \) and the associated transfer function \( U(f) \) were defined. We shall assume now that the scatterers have the same characteristics but that their strength is random. This is taken into account by expressing the scattering transfer function \( U(f) \) in the form \( \alpha U_0(f) \), where \( U_0(f) \) is a deterministic function of the frequency \( f \) and where \( \alpha \) is a random real variable. The statistical properties of \( \alpha \) are assumed to be such that its mean value denoted \( \bar{\alpha} \) is zero:

\[
\bar{\alpha} = 0. \tag{2.20}
\]

The expression for the spectrum of a single target becomes:

\[
E(r,f) = \alpha M(f) U_0(f) H^2(r,f) \tag{2.21}
\]

and:

\[
S(r,f) = M(f) U_0(f) H^2(r,f). \tag{2.22}
\]

Then \( s(r,t) \) denotes the inverse Fourier transform of \( S(r,f) \) and allows rewriting the target response to be rewritten

\[
e(r,t) = \alpha \cdot s(r,t) \tag{2.23}
\]

which is a very meaningful form because in this way the stochastic part of the process is described by the random variables \( \alpha \) and \( r \), while the deterministic part relies completely on the function \( s(r,t) \). With the linearity hypothesis expressed in sec. 1.1, the signal returned by a collection of scatterers numbered from 1 to \( N \), the \( i \)-th being located in \( r_i \) and having a scattering strength \( \alpha_i \), reads

\[
\sum_{i=1}^{i=N} \alpha_i \cdot s(r_i,t).
\]

Let us now examine the assumption: \( \bar{\alpha} = 0 \). This assumption holds if the sign of \( \alpha \) is random with the same probability for + and −. In the classical theory, scattering is due to discontinuities in the propagation medium, that is to say to local discrepancies in compressibility or density and the sign of the scatterer depends on the sign of the discrepancy. Therefore the condition \( \bar{\alpha} = 0 \) is found in a medium where differences in compressibility or density appear randomly above or below the average value. However, in tissue the origin of these discrepancies can be attributed to the biological structure. It is also possible that the sign of the discrepancies is the same for all
2.9. The notion of complex signal

The notion of complex signal is very useful and this brief section reviews its essential properties. Commonly, a real monochromatic signal \( \cos (\omega t) \) is represented in complex form as \( \exp (i\omega t) \). It is possible to extend this transformation to a case where the signal is not monochromatic but where the spectrum covers a frequency band, and in this way to again associate the real signal \( s(t) \) with a complex signal \( s^+(t) \). A very easy way of describing this transformation is to use the frequency domain.

A real signal \( s(t) \) has a Fourier transform \( S(f) \) with the property \( S(-f) = S^*(f) \). It should therefore be noted that the information carried by the negative frequencies is redundant but allows one to ‘build’ the real signal by adding a positive and a negative frequency signal. The analytical signal is derived from the inverse Fourier transform of the spectrum of the real signal after suppression of the negative frequencies and then multiplied by two. In this way a complex signal \( s^+(t) \) with the following properties is obtained:

- \( s^+(t) \) has the same spectrum as \( s(t) \) for positive frequencies;
- \( s^+(t) \) contains no negative frequencies;
- \( s^+(t) \) easily leads to \( s(t) \) with \( s(t) = \text{Re}(s^+(t)) \).

Two very useful properties of the analytical signal are worth noting:
- \( s^+(t) \) supplies the envelope \( A(t) \) of the real signal
  \[ A(t) = |s^+(t)|; \]  
  \[ (2.24) \]
- \( s^+(t) \) gives unequivocally the signal phase denoted \( \phi(t) \) defined as the angle between \( s^+(t) \) and the real axis.

It is often useful to represent the signal by amplitude and phase according to

\[ s^+(t) = A(t) \exp (i \phi(t)). \]  
\[ (2.25) \]
The use of the analytical signal is so rewarding that it will consistently be used from now on and will no longer be distinguished from \( s \) by the notation \( s^+(t) \).

3. Time-frequency energy distributions

In Signal Theory there are several possible representations of a signal. The most immediate describes the evolution in time of a physical magnitude that describes a phenomenon. For the study of stationary signals the Fourier Transform gives a frequency spectrum which constitutes an alternative and complete representation but excludes the time parameter. In the study of nonstationary signals, one really needs new processing methods which allow a representation of the signal both in its time and frequency span. The ultrasound signal is a striking example of a nonstationary process: due to the propagation phenomenon, the latest echoes are those coming from the deepest-lying tissue; hence they have also undergone the strongest propagation effects so that their spectral content clearly differs from earlier echoes. For this reason, it is essential to give a sense to the notion of 'instantaneous spectrum'.

Since the first paper by Ville \(^2\) it has taken some time to realize that there was no unique solution to this problem. On the contrary, several definitions have been suggested, each having its own merits and presenting a certain number of attractive properties. Successive attempts at unifying the various representations have been presented and the relationships between the different representations have been gradually clarified. In this chapter we shall deal mainly with the Rihaczek representation and with the one known as the 'spectrogram'. The Wigner-Ville representation is also briefly discussed in the last section. It should be noted that all these representations are a description of energy and that they therefore refer to energy density.

In tissue characterization tasks, we are often interested in the temporal evolution of the power, the frequency or the bandwidth of the signal. Any time-frequency energy distribution allows the precise definition and computation of such quantities which yield a 'summary' of the spectral evolution, the value and usefulness of which depend on a proper choice of the distribution.

3.1. The spectrogram

The spectrogram is the most intuitive representation of the energy evolution of a signal both in time and frequency \(^3,4\). An analysis window denoted \( w(t) \) is used to isolate a time section of the signal. A Fourier transform then gives the spectral components of the selected part of the signal. As the
spectrogram depends on the window \( w \), it is denoted here \( P_w(t, f) \) and defined by

\[
P_w(t, f) = \sqrt{\int_{-\infty}^{+\infty} s(t') w(t' - t) \exp(-2i \pi ft') dt'}^2. \tag{3.1}
\]

A temporal convolution can be recognized here

\[
\int_{-\infty}^{+\infty} s(t') w(t' - t) \exp(-2i \pi ft') dt' = (s(t) \exp(-2i \pi ft)) \ast w(-t) \tag{3.2}
\]

and considering \( w(-t) \) as the impulse response of a low-pass filter, the spectrogram appears as the power at the output of this filter when it is driven by the signal \( s(t) \cdot \exp(-2i \pi ft) \), i.e. the signal obtained by beating with a signal of frequency \( f \). This can be interpreted as follows: to observe the time evolution of the energy contents of the signal at frequency \( f \), the signal is heterodyned so that the frequency band in question faces the low-pass filter band (therefore centered in \( f = 0 \)). The low-pass filter then shows only one frequency band. Its power output is the spectrogram. The resolution obtained in both time and frequency can then be discussed. The frequency resolution is limited by the bandwidth of the filter defined by \( w(-t) \), noted \( \Delta F_w \). The time resolution is limited by the time duration of the impulse response of the filter or equivalently by the temporal extent of the window, denoted \( \Delta T_w \). However, it is known to be physically impossible to simultaneously make \( \Delta F_w \) and \( \Delta T_w \) arbitrarily small because 'window' type functions (low-pass filters) satisfy

\[
\Delta F_w \cdot \Delta T_w = 1. \tag{3.3}
\]

In other words, any attempt to modify a given window to increase frequency resolution (for example) leads to a loss of resolution in time and vice-versa.

The limits of resolution inherent in spectrograms take a special form for the ultrasound signal. In fact, it is made up of a sum of coherent echoes, each of these having a time duration \( \Delta T_e \) and a spectral width \( \Delta F_e \), broadly determined by the electromechanical system. If, as usual, echoes are generated by a transducer excited in a single mode, we have

\[
\Delta F_e \cdot \Delta T_e = 1. \tag{3.4}
\]
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The spectral width of the entire ultrasound signal is that of each individual echo, namely $\Delta F_e$. Therefore, to analyse the signal over $n$ frequency bands, we need a filter (the window) with a spectral width $\Delta F_w = \frac{\Delta F_e}{n}$. The time resolution of the spectrogram is therefore imposed by

$$\Delta T_w \approx n \cdot \Delta T_e.$$  

(3.5)

It is worth repeating here two important conclusions:

i) For a spectrogram to provide a useful spectral resolution in $n$ independent bands, the time resolution has to be at least equal to $n$ times the span of an isolated echo.

ii) The limits of spectrogram resolution imply that it is impossible to analyse an isolated echo. This results directly from (i) and is due essentially to the fact that $\Delta F_e \cdot \Delta T_e \approx 1$. It would be meaningless, for example, to divide the echo into three time zones: beginning, middle and end of echo, and then to carry out a frequency analysis, as the resolution limit in spectrogram frequency becomes superior to the spectral width itself of the signal under analysis.

As will be shown in the last chapter, devoted to an analysis of some tissue characterization methods, most of the algorithms published up to now rely (explicitly or not) on the spectrogram.

3.2. The Rihaczek distribution

The use of the spectrogram is a first, obvious approach to determine the time-frequency energy density but it raises two problems related to window choice and resolution. A window is needed for the spectrogram, but in the absence of absolute criteria for its choice it could happen that the ‘real’ distribution of energy is lost and only approximate or deformed results are observed. Whatever window is chosen, it presupposes a time-frequency resolution limit. Is this limit in itself an ultimate or good limit and does the use of the window ‘blur’ a ‘real’ distribution of energy in time and frequency? The Rihaczek distribution$^5$ is an attempt to answer these questions. There are several ways of deriving the definition suggested by Rihaczek. In this section we recall his approach based on the notion of interactive energy.

The interactive energy between two analytical signals $s_1(t)$ and $s_2(t)$ during time interval $t_1 < t < t_2$ is classically defined by

$$E = \int_{t_1}^{t_2} s_1(t) s_2(t) \, dt.$$  

(3.6)
The Rihaczek distribution for a signal $s$ can be obtained from the interaction energy between the two signals:
- the signal $s$ truncated in time between $t$ and $t + dt$;
- the signal $s$ truncated in frequency between $f$ and $f + df$.

As we have

$$s(t) = \int_{-\infty}^{+\infty} S(f) \exp(2i\pi ft) df, \quad (3.7)$$

this frequency truncation on $s$ results in: $S(f) \exp(2i\pi ft) df$. The interactive energy is therefore

$$\int_{t}^{t+dt} s(t) \cdot \left\{ S(f) \exp(2i\pi ft) df \right\}^* dt = s(t) S^*(f) \exp(-2i\pi ft) dt df$$

and the density of time-frequency energy is

$$\rho(t, f) = s(t) S^*(f) \exp(-2i\pi ft). \quad (3.8)$$

The expression 'time-frequency density' will be abbreviated to TFD.

It can easily be verified that $\rho(t, f)$ has the properties aimed at:

i) If the signal to be analysed is time-shifted or frequency-shifted, then its TFD follows the corresponding shift.

ii) The integral of $\rho(t, f)$ for all the frequencies again yields the time density of energy:

$$\int_{-\infty}^{+\infty} \rho(t, f) df = |s(t)|^2. \quad (3.9)$$

iii) The integral of $\rho(t, f)$ in time yields the spectral density of energy:

$$\int_{-\infty}^{+\infty} \rho(t, f) dt = |S(f)|^2. \quad (3.10)$$

As seen further on, it is very useful to ‘summarize’ the spectrum supplied
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by the Rihaczek distribution at a given moment by its frequency moment data, or the quantities defined by

\[ m_n(t) = \int_{-\infty}^{+\infty} \rho(t, f) f^n df. \] (3.11)

Since in the Fourier domain a temporal derivation turns into a multiplication by \(2i\pi f\), an interesting expression of these moments is easily derived:

\[ m_n(t) = \left(\frac{i}{2\pi}\right)^n \cdot s(t) s^{*n}(t), \] (3.12)

using the product of the signal and the \(n\)th derivative of its conjugate. It should be noted that in this expression only values of the signal and of its derivatives at time \(t\) appear. This property thus ensures that the distribution moments are actually 'instantaneous'.

The moments of the distribution are directly linked to meaningful quantities. From eq. (3.9) it can be seen that the moment of order zero is just the signal power. That is why we will often prefer the more expressive notation:

\[ p(t) = m_0(t). \] (3.13)

The instantaneous frequency denoted \(\nu(t)\) is defined as the real part of the distribution frequency centroid at time \(t\), so that it can be expressed as the real part of the ratio of the first two moments:

\[ \nu(t) = \frac{\text{Re}(m_1(t))}{m_0(t)}. \] (3.14)

Expressing \(\nu(t)\) as a function of amplitude and phase is quite instructive. From eqs. (3.12) and (2.25) we have

\[ \frac{m_1}{m_0} = \frac{i}{2\pi} \cdot \left(\frac{s'}{s}\right)^* = \frac{i}{2\pi} \left(\frac{d \ln(s)}{dt}\right)^* = \frac{1}{2\pi} \frac{d\phi}{dt} + \frac{i}{2\pi} \frac{A'}{A}. \]

\[ \nu(t) = \frac{1}{2\pi} \frac{d\phi}{dt}. \] (3.15)
The instantaneous frequency (3.14) is quite simply the phase derivative. However, the imaginary part of \( m_1/m_0 \) is not meaningless, as the following example shows. Let us consider a sine-wave modulated in amplitude:

\[
s(t) = (1 + a \cos 2\pi f_1 t) \cdot \exp(2i \pi f_0 t).
\]

Its instantaneous frequency is simply \( \nu(t) = f_0 \) whereas the imaginary part \( \text{Im} (m_1/m_0) \) is a term oscillating at frequency \( f_1 \). Thus, considering only \( \text{Re} (m_1/m_0) \), the modulation \( f_0 - f_1 \) and \( f_0 + f_1 \) centered around \( f_0 \) is not revealed. It is in \( \text{Im} (m_1/m_0) \) that the presence of the modulation is observed.

### 3.3. Relationship between the spectrogram and the Rihaczek density

We shall define now the relation which links the spectrogram to the Rihaczek distribution, and then deduce the relations between the moments.

The spectrogram \( P_w(t, f) \) of the signal \( s(t) \) uses the time window \( w(t) \). We note the Rihaczek distributions associated to \( s \) and \( w \), as \( \rho_s \) and \( \rho_w \) respectively. We have\(^6\)

\[
P_w(t, f) = \int_{-\infty}^{+\infty} \rho_s(t', f') \rho_w(t' - t, f' - f) \, dt' \, df', \text{ i.e.} \quad (3.16)
\]

\[
P_w(t, f) = \rho_s(t, f) * \rho_w(-t, f). \quad (3.17)
\]

The spectrogram is deduced therefore from the Rihaczek distribution by a convolution in the time-frequency plane with the Rihaczek distribution related to the window. This relation evidences the double effect of the window: it smooths the TFD by a convolution to arrive at the spectrogram but it always provides a real and positive magnitude. The loss of resolution is the price to pay to obtain a real and positive energy density.

The relations between moments can now be derived from eq. (3.17). Let us call \( m_n^+ \) \( (t) \) the frequency moment of the order \( n \) evaluated with the spectrogram:

\[
m_n^+(t) = \int_{-\infty}^{+\infty} P_w(t, f) f^n \, df. \quad (3.18)
\]

The relation (3.17) after integration, links \( m_n^+ \) to the frequency moments of the window and signal calculated with the Rihaczek distribution and de-
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noted respectively \( m^w_n(t) \) and \( m^s_n(t) \). A variable change for \( f \) yields the binomial coefficients \( C_n^p \) and after some algebra we get:

\[
m^+_n(t) = \sum_{p=0}^{p=n} C_n^p m^s_p(t) \ast m^w_{n-p}(-t).
\] (3.19)

Let us elaborate this relation with the moment of the order zero, that is power. The last relation yields for \( n=0 \).

\[
m^+_0(t) = m^s_0(t) \ast m^w_0(-t) = |s(t)|^2 \ast w^2(-t).
\] (3.20)

Therefore the spectrogram yields an energy which is precisely the signal power, given instantaneously by \( |s(t)|^2 \), and smoothed by a low-pass filter with impulse response \( w^2(-t) \).

Let us now consider the running-time frequency defined on the spectrogram which is frequently called ‘centroid’, a real positive quantity defined as:

\[
f^+(t) = \frac{m^+_1(t)}{m^+_0(t)}.
\] (3.21)

The relationship between the centroid and the instantaneous frequency is derived from eq. (3.19) which is simplified by noting that \( m^s_0 \) and \( m^w_0 \) are pure real numbers while \( m^s_1 \) is a pure imaginary one:

\[
f^+(t) = \frac{\text{Re}(m^s_1(t) \ast m^w_0(-t))}{m^s_0(t) \ast m^w_0(-t)}.
\] (3.22)

Using the instantaneous amplitude and phase of the signal to express its Rehaczek moments (eq. 3.13 and 3.14) the centroid frequency is:

\[
f^+(t) = \frac{A^2 \nu \ast m^w_0(-t)}{A^2 \ast m^w_0(-t)}.
\] (3.23)

This formula is important and deserves some further comment. Using \( m^w_0(-t) = w^2(-t) \) it can be rewritten in a more expressive and explicit form as:
Examination of this relation shows that the centroid gives a frequency \( f^c \) which is the moving weighted average over a certain window \( (w^2(-t)) \) of the instantaneous frequency, the weighting factor being the power \( A^2(t) \). The fact that the centroid averages the instantaneous frequency proportionally to the power that carries this frequency constitutes a valuable property of the spectrogram, as will be shown in sec. 4.4.

3.4. The Wigner-Ville distribution

J. Ville proposed \(^2\) in 1948 an energy distribution widely known as the Wigner-Ville distribution, defined by

\[
\rho_V(t, f) = \int_{-\infty}^{+\infty} s(t + \tau/2) s^*(t - \tau/2) \exp(-2i\pi f \tau) d\tau. \quad (3.25)
\]

This distribution satisfies the 'expected' properties (sec. 3.2). By performing the integration in the frequency domain, we have obtained, as with the Rihaczek distribution, an 'instantaneous' expression for the frequency moment of the Wigner-Ville distribution:

\[
m_n(t) = \frac{1}{(2i\pi)^n} \left[ \frac{d^n}{d\tau^n} (s(t + \tau/2) s^*(t - \tau/2)) \right]_{\tau=0}. \quad (3.26)
\]

Using this relation, instantaneous power and frequency can be expressed as functions of instantaneous amplitude and phase. They appear to have the same expression as for the Rihaczek distribution. A simple formula evidences the close relation between the Ville and Rihaczek distributions\(^2,5\):

\[
\rho_V(t, f) = \rho_R(t, f) * \exp(4i\pi ft). \quad (3.27)
\]

Considering the widespread use of the Wigner-Ville distribution we have given its definition here but, in our case, where the main interest is in the
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study of moments, this distribution does not supply us with any additional information. Moreover, the Rihaczek distribution presents the advantage of a simpler mathematical expression, allowing us, as shown in the following chapter, to complete easily a certain number of statistical evaluations. These two reasons have led us to prefer the Rihaczek distribution to the Wigner-Ville one.

4. Statistics on time-frequency distribution

4.1. Notations

In sec. 3 we presented the properties of energy distributions in time and frequency without specifying the considered signals. In this chapter, using statistical hypotheses and the model defined in sec. 2, we shall deal with the case of an ultrasound signal considered as a random non-stationary signal. We shall calculate expectations of the Rihaczek distribution and of its moments and explore the correlation between amplitude and frequency. The notations to be followed in this chapter are summarized first below.

- \( \xi \) is the random variable describing a spatial distribution of scatterers and their reflectivity. In this way for a distribution of \( N \) scatterers, \( \xi \) represents \( N \) vectors of positions \( r_i \) with \( i = 1, \ldots, N \) and \( N \) values of reflectivity \( \alpha_i \) with \( i = 1, \ldots, N \).
- The signal resulting from a distribution \( \xi = (r_i, \alpha_i) \) with \( i = 1, \ldots, N \) is labelled

\[
s(\xi,t) = \sum_{i=1}^{N} \alpha_i s(r_i, t), \quad (4.1)
\]

and its Rihaczek distribution

\[
\rho(\xi,t,f) = s(\xi,t) S^*(\xi,f) \exp(-2i \pi f t). \quad (4.2)
\]

The moments of this distribution are

\[
m_n(\xi,t) = \int \rho(\xi,t,f) f^n \, df. \quad (4.3)
\]

The deterministic description of the system relies on the point response \( s(r,t) \) defined in sec. 2.8. Its Rihaczek distribution is noted \( \rho(r,t,f) \):

\[
\rho(r,t,f) = s(r,t) S^*(r,f) \exp(-2i \pi f t), \quad (4.4)
\]
its moment of order \( n \) defined on this distribution, \( m_n(r,t) \),

\[
m_n(r,t) = \int \rho(r,t,f) f^n \, df. \tag{4.5}
\]

With these notations, the difference between the random and the deterministic functions is evidenced by the type of the first variable: the global random variable \( \xi \) for the random functions, the spatial variable \( r \) for the point functions depending on target location. The aim of statistical evaluation is to compute ensemble averages, in other words: 'to make \( \xi \) disappear'. The expected values for \( \rho(\xi,t,f), m_n(\xi,t) \) etc... will then be computed by considering spatial integrals of \( \rho(r,t,f), m_n(r,t) \) etc... These integrals can be seen as spatial averages, in other words, 'we make \( r \) disappear'. Finally the whole process is to express statistical averages as a function of spatial deterministic averages. The statistical average of a random function \( X(\xi) \) of \( \xi \) is denoted \( E(X) \) or \( \bar{X} \). The spatial average of a point function \( X(r) \) over \( \Omega \) the half space in front of the transducer is denoted \( <X> \) and defined as:

\[
<X> = \int \int \int_X (r) \, dr. \tag{4.6}
\]

Finally, as a non-stationary context is always there, the explicit temporal dependance is often dropped. For instance, at any instant \( t \) the notation \( <m_n> \) is for:

\[
<m_n> = <m_n>(t) = \int \int m_n(r,t) \, dr. \tag{4.7}
\]

4.2. Expected value of the Rihaczek distribution

Let us, in accordance with the assumptions made in sec. 2.8, consider \( N \) targets located in a given volume \( V \). The position of the \( i \)-th target is labelled \( r_i \) and its reflectivity coefficient is labelled \( \alpha_i \) so that the contribution of this target to the time signal is \( \alpha_i \cdot s(r_i,t) \) and its contribution to the spectrum is \( \alpha_i \cdot S(r_i,f) \). For a given distribution \( \xi \) we can observe the signal

\[
s(\xi,t) = \sum_{i=1}^{N} \alpha_i \cdot s(r_i,t), \tag{4.8}
\]
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and the spectrum reads

\[ S(\xi, f) = \sum_{i=1}^{N} \alpha_i S(r_i, f), \quad (4.9) \]

and therefore the distribution of energy is

\[ \rho(\xi, t, f) = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* s(r_i, t) S^*(r_j, f) \exp(-2i \pi f t). \quad (4.10) \]

Because the statistical properties of the scattering strength have been assumed to be independent of position we have:

\[
E(\alpha_i \alpha_j^* s(r_i, t) S^*(r_j, f) \exp(-2i \pi f t)) \\
= E(\alpha_i \alpha_j^*) E(s(r_i, t) S^*(r_j, f) \exp(-2i \pi f t))
\]

and

\[
\tilde{\rho}(t, f) = E(\rho(\xi, t, f)) \\
= \sum_{i=1}^{N} \sum_{j=1}^{N} E(\alpha_i \alpha_j^*) \cdot E(s(r_i, t) S^*(r_j, f) \exp(-2i \pi f t)). \quad (4.11)
\]

Within this double sum a distinction between diagonal and non-diagonal terms should be made. The latter have a zero mathematical expectation value since the scattering strengths are zero mean and uncorrelated with each other: \( E(\alpha_i \alpha_j^*) = E(\alpha_i) \cdot E(\alpha_j^*) = 0 \). There remain, therefore, within the sum, only \( N \) non-zero diagonal terms and, because all the scatterers have the same statistical properties:

\[
\tilde{\rho}(t, f) = N E(\alpha_i \alpha_i^*) E(s(r_i, t) S^*(r_i, f) \exp(-2i \pi f t)) \\
= N \overline{\alpha^2} E(\rho(r, t, f)). \quad (4.12)
\]

This straightforward result has a simple interpretation: the energy of interaction between \( N \) echoes is zero in statistical average but there remains \( N \) times the mean energy distribution of a single scatterer. This is a classical result when dealing with incoherent summations of waves (on average, energies and not amplitudes are to be added) extended here to a non-stationary case. As the density of the probability of the position is uniform in
a volume \( V \), it equals \( 1/V \) and:

\[
E(\rho(r,t,f)) = \frac{1}{V} \int \int \int \rho(r,t,f) \, dr.
\]  
(4.13)

By stretching the volume \( V \) over the complete frontal half-space \( \Omega \) and denoting \( n = N/V \), we obtain the scatterer density:

\[
\bar{\rho}(t,f) = n \alpha^2 \int \int \int \rho(r,t,f) \, dr = n \alpha^2 <\rho(r,t,f)>.
\]  
(4.14)

It should be noted that (4.2.7) is obtained without 'randomizing' \( N \). It is not necessary to give \( N \) a Poisson distribution since (4.2.7) is obtained by stretching the domain of integration to infinity keeping a constant density \( n = N/V \): such a procedure gives an implicit Poisson distribution of parameter \( n \).

In the next section we present a transform of this expression that gives more physical insight and allows a thorough discussion leading to the definition of the diffraction filter. Moreover, this will make it clear that the mean energy distribution is actually real and positive and consequently its frequency moments as well.

The expectation value of a moment \( \bar{m}_q(t) \) is defined by

\[
\bar{m}_q(t) = E(m_q(\xi,t)) = E\left( \int \rho(\xi,t,f) f^q \, df \right),
\]  
(4.15)

but because of linearity we have

\[
\bar{m}_q(t) = \int E(\rho(\xi,t,f)) f^q \, df = \int \bar{\rho}(t,f) f^q \, df = n \alpha^2 \int \int \int m_q(r,t) \, dr
\]  
(4.16)

and finally

\[
\bar{m}_q(t) = n \alpha^2 <m_q(r,t)>.
\]  
(4.17)

According to our assumptions, the statistical average is simply the spatial average multiplied by the density of targets and mean scattering intensity. This very simple result is derived thanks to the vanishing average interactive energy between targets.
4.3. A physical expression of volume integrals

The expressions obtained in the previous sections may appear difficult to grasp. One of the obstacles to their understanding is that we include target responses at constant $t$, with integration over the variable of position, where as observing time dependence, for a given position $r$ is more familiar. A slight approximation on $s(r,t)$ yields alternative expressions which make it easier to understand their physical meaning. The general form of the integrals dealt with is $G(t) = \int \int \int g(r,t) \, dr$, $r$ varying throughout the front half-space while $t$ is fixed. However, the functions to be integrated are scatterer signals, derivatives of these signals and their products. These signals have a finite duration. Therefore, in $G(t)$ the only appreciable contribution to the integral comes from targets such as $|r| = \frac{1}{2}ct$, which corresponds to a half-spherical thin surface centered on the transducer with poorly defined edges and with a thickness roughly equal to the spatial ‘length’ of the ultrasonic pulse.

An interesting approximation follows from the statement that for a determined direction of $r$ and throughout the full thickness of this film, the target signals are simply time-shifted. In polar coordinates, hence $r: (r, \theta, \phi, t)$, this property can be expressed as

$$s(r + \delta r, \theta, \phi, t) \approx s(r, \theta, \phi, t - \frac{\delta r}{c}). \quad (4.18)$$

This approximated result is valid for all products of $s$ and of its derivatives and therefore, in general, for all the functions to be considered:

$$g(r + \delta r, \theta, \phi, t) \approx g(r, \theta, \phi, t - \frac{\delta r}{c}). \quad (4.19)$$

It is then possible to change the space variable into a time variable:

$$G(t) = \int \int \int g(r, t) \, dr = \int \int \int g(r, \theta, \phi, t) \, r^2 \sin \theta \, dr \, d\theta \, d\phi. \quad (4.20)$$

As $|r|$ has a small variation around the value $r_0 = \frac{1}{2}ct$, a second approximation can be made by considering the term in $r^2$ in eq. (4.20) as a constant:
Changing the variables according to \( r = r_0 + \frac{ct}{2} = \frac{ct}{2} + \frac{cT}{2} \),

\[
\int_{r=0}^{\infty} g(r, \theta, \phi, t) \, dr = \frac{c}{2} \int_{-t}^{\infty} g(r_0, \theta, \phi, \tau) \, d\tau.
\]

Hence,

\[
G(t) \approx \int \int \int_{\theta, \phi}^{\infty} r_0^2 \sin\theta \left( \frac{c}{2} \int_{-\infty}^{+\infty} g(r_0, \theta, \phi, \tau) \, d\tau \right) \, d\theta \, d\phi
\]

and finally:

\[
G(t) = \frac{c}{2} \int \int \left( \int_{t}^{+\infty} g(r, \tau) \, d\tau \right) \, dr
\]

where \( \Sigma(t) \) designates the isochronal surface at the moment \( t \), that is to say all points \( r \) such that \( |r| = \frac{1}{2}ct \).

This transformation is interesting for several reasons:

– A physical sense is frequently found to the time integral \( \int_{-\infty}^{+\infty} g(r, \tau) \, d\tau \). For example, if \( g(r, \tau) = p(r, \tau) \) then the time integral is the total energy of the response from the scatterer located in \( r \).

– Once the time integral is known, the spatial integration domain has a clear physical meaning: it is the isochronal surface.

– Some analytical calculations can be carried out on this second expression as shown later.

– Several numerical calculations require less computing time and memory space.

It is possible to apply the same approximation to the expression of \( E(\rho) = \tilde{\rho}(t, f) \) by using the property of the Rihaczek distribution that the time
translation of the signal corresponds to the same time translation of the dis-
tribution. Therefore, as (from eq. 3.10):

\[ \int_{t = -\infty}^{t = +\infty} \rho(r, t, f) \, dt = |S(r, f)|^2, \quad (4.25) \]

the equation 4.24 now reads:

\[ \tilde{\rho}(t, f) = n \frac{c}{2} \alpha^2 \int \int_{r \in \Sigma(t)} |S(r, f)|^2 \, dr. \quad (4.26) \]

To understand the distribution of energy in frequency at a given moment, it suffices to integrate over all possible locations on the isochronal surface the power spectrum of the response from these locations. This expression is rather fundamental and will be used again in sec. 5 to define the diffraction filter and in sec. 6 to show how diffraction effects can be corrected.

The latest expression has another value: it shows directly that the mean Rihaczek distribution is real-valued and even positive since all factors in \((4.26)\) are real positive numbers. As a direct consequence, all its moments \(\tilde{m}_q(t)\) are also real and positive.

4.4. Power-frequency correlation, negative frequency

In this section we do not refer to a Rihaczek distribution but rather pro-
vide a simple analysis of the variance of the instantaneous frequency. In or-
der not to be confused by other effects the simplest model is assumed here, i.e. the limit situation where all the individual echoes have the same instan-
taneous frequency throughout the whole length of the echo (i.e. each echo is a sine-wave modulated in amplitude) so that only the interference be-
tween individual echoes is a source of frequency variance. All the echoes are also assumed to have the same temporal envelope, which means that an echo from the \(k\)-th scatterer is now characterized only by its time position \(t_k\) and is expressed as

\[ s(t_k, t) = a(t - t_k) \exp (2i \pi f_0(t - t_k)). \quad (4.27) \]

The Fresnel construction is familiar to anyone who has studied vibrations. When the analytical signal is used, the Fresnel plane is identified simply with the complex plane. In this plane, the path of an isolated echo is a spiral with
an increasing and then decreasing radius. More precisely, since the echo is readily expressed with amplitude and phase in sec. 4.27, its path in the complex plane is a curve expressed in the polar coordinate system \((d, \theta)\) simply by

\[
d(t) = a(t - t_k), \quad \theta(t) = 2\pi f_0 (t - t_k). \tag{4.28}
\]

To show more clearly the effect of interference, the observer should be placed in a rotating position. If the angular speed of this rotating position is \(\omega = 2\pi f_0\), then all the echoes that have a constant frequency \(f_0\) appear to this observer as having a constant phase. The path of an individual echo then describes a straight line in the rotating Fresnel plane and the trajectory of (4.27) therefore becomes

\[
d(t) = a(t - t_k) \quad \theta(t) = \theta_k = -2\pi f_0 t_k = \text{constant.} \tag{4.29}
\]

The coherent sum of \(N\) echoes of the signal is

\[
s(t) = \sum_{k=1}^{N} a(t - t_k) \exp(2i\pi f_0(t - t_k)). \tag{4.30}
\]

Its path in the complex rotating plane, the end of the sum vector, is described by the complex number

\[
Z = \sum_{k=1}^{N} a(t - t_k) \exp(-2i\pi f_0 t_k). \tag{4.31}
\]

Each element of this sum describes, with a certain angle equal to \(\theta_k = -2\pi f_0 t_k\), a straight trajectory defined by \(a(t - t_k)\). The sum of all these vectors traces in the plane an involved trajectory which depends largely on interference conditions, i.e. on the distribution of phases determined by the \(t_k\). The norm of the vector sum \(Z\) is also the amplitude \(A(t)\) of the sum signal, and its angle \(\theta(t)\) is the signal phase in the rotating reference. The time derivative of this angle, \(\dot{\theta}(t)\), defines the angular velocity of \(Z\) and \(\Delta f(t) = 2\pi \dot{\theta}(t)\) is the signal frequency in the rotating reference. In the fixed reference the signal frequency is simply \(f(t) = f_0 + \Delta f(t)\). In this way we only have to consider the angular velocity of the sum of the echoes in the rotating reference to analyse the departures of the signal frequency from the value \(f_0\). If the distribution of delays \(t_k\) is considered to be random with a uniform
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probability density, then the factors \( \exp(-2i \pi f_0 t_k) \) are statistically isotropic, in other words there is no privileged direction in the rotating reference. Hence, the value of \( \Delta f(t) \) can be negative or positive with the same probability. Therefore, the average value of \( \Delta f(t) \) is zero and the mean frequency value of the signal is \( f_0 \). In this case, the interference introduces no bias on the average frequency which is in fact the frequency of individual echoes. To reach our conclusion, it suffices to remember that the angular velocity of a point is equal to the linear tangential velocity divided by the distance of the point from the centre of rotation. So when \( Z \) passes close to the origin, even if its linear velocity is low, it shows a higher angular velocity and then a high value of \( \Delta f \) is observed.

In the time evolution of the signal, the departures in the instantaneous frequency from the average instantaneous frequency are then correlated with off-peak signal power and the conclusion is that a local weakening in energy of a coherent sum of echoes indicates that the instantaneous frequency at that moment more likely departs from its average value. This property is illustrated in fig. 3. The upper trace represents a simulated ultrasound signal as a superposition of strictly identical Gaussian pulses spread over time in a random manner. The second trace represents the power of this same signal. There are three instants where the received power is very weak (the curve then meets the axis). The lower trace shows variations in instantaneous frequency around its average value (represented by a horizontal asymptote) of

![Fig. 3. A simulated echographic response s, its instantaneous power P and frequency f.](image-url)
2.5 Mhz (which is also the central frequency of the Gaussian pulses). Four strong deviations in instantaneous frequency are observed which coincide precisely with the four values of low power.

The instantaneous frequency of the signal varies rapidly in time: in order to use it in tissue characterization algorithms, a smoothing of this function is required. This smoothing is, mathematically, a time convolution by a low-pass filter. Our analysis shows that the efficiency of this smoothing can be increased by weighting over the smoothing time interval (i.e. over the duration of the low-pass filter impulse response) the frequency values by power (i.e. the square of the amplitude), in order to rely more heavily on those frequencies which are carried by high power. This throws light on the final remark of sec. 3.3 where the centroid, (i.e. the spectrogram mean frequency) appeared as the result of a smoothing of the instantaneous frequency proportionally weighted by power. That is why the spectrogram is a good tool for the measurement of frequency with ultrasound-type signals (coherent sum of pulses).

5. Diffraction filter

5.1. Mean distribution of energy in time and frequency

The expected distribution of energy in time and frequency for a scattering system described by the space-varying transfer function $S(r, f)$ (eq. 4.26) was derived in sec. 4. Combining this statistical result with the physical model for $S(r, f)$ obtained in sec. 2 (eq. 2.12 and eqs 2.22) gives

$$E(\rho(t, f)) = \frac{n c}{2} |M(f)| U(f) \exp(-2a(f)c t) \int \int |H_0(r, f)|^4 dr. \quad (5.1)$$

This important result shows how the energy distribution in time and frequency is related to the physics of ultrasound. With a view to estimating attenuation, it suggests the separation of all the intervening phenomena into three groups, corresponding to the factorization of $E(\rho)$ in three terms, as briefly discussed below.

- The time (or depth) invariant effects are regrouped in the term $n c/2 |M(f)| U(f)$. The quantities $U(f)$, $n$ and $c$ depend on the scattering medium, while $M(f)$ is 'system-dependent'. This term is not known exactly but under our hypothesis its more important characteristic is that it is time-independent. That is why tissue characterization (TC) algorithms rely on spectra comparison or observation of spectrum evolution, thus
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Fig. 4. Diffraction filter multiplied by \( t^2 f^2 \) for a plane transducer (radius = 5 mm).

eliminating the influence of unknown time-invariant transfer functions.
- The time-dependent attenuation effect is modelled by \( \exp(-2a(f)ct) \). Here \( a(f) \) is the unknown function to be estimated. Thanks to a time-frequency representation it is clearly factorized in the expression of \( E(p) \).
- The time dependent diffraction effect is expressed by the integral on the isochronous surface of the one-way diffraction transfer function modulus to power 4. We thus define the diffraction filter \( D(t, f) \) as

\[
D(t, f) = \int \int_{r \in \Sigma(t)} |H_0(r, f)|^4 dr.
\] (5.2)

Hence, the diffraction effect on energy distribution can be separated from any other physical effect. It varies in time, like the attenuation effect, so that it must be accurately characterized to remove its contribution to spectral changes; this is the necessary condition for estimating without a diffraction artefact an unbiased value of attenuation.

The diffraction filter of a plane circular transducer of radius 5 mm has been computed, using the Stepanishen formula \(^1\) (see sec. 2.3) for the diffraction impulse response. An easy visualization of the phenomenon is to plot the values of \( t^2 f^2 D(t, f) \), as in fig. 4 (otherwise the perspective plot would not have been readable since \( D(t, f) \) exhibits a strong maximum for \( t=0, f=0 \).

5.2. Diffraction filtering

The separation of the diffraction from other’s phenomena has been made possible through three conditions: only its influence on energy is considered, the energy evolution is described by a distribution in time and frequency simultaneously, and the effect on the mean distribution is considered (statistical expectation). Here are some comments about these three points.
The diffraction filter is defined by its action on the signal energy: at any time $t$, $D(t, f)$ as a function of frequency is a power spectrum. Without additional constraints (such as the minimum phase condition) a running time transfer function or impulse response cannot be deduced from the values of $D(t, f)$: the mean diffraction is expressed more directly in the energy domain.

As emphasized in the first chapter, the non-stationarity of the echographic signal is partly due to diffraction. Hence only a time-frequency representation simply describes the diffraction effect. However, if the diffraction filter were separable into the product of a function of time by a function of frequency it could be simply compensated (or represented) by the action of an invariant filter (a function of $f$) followed by a time-varying gain control (a function of $t$). As will be shown in the next sections this is not the case and the diffraction effect could be called 'strictly' non-stationary.

The last point is that $D(t, f)$ is a statistical filter. By this we mean that the effect of diffraction on a single echographic signal cannot be deduced from $D(t, f)$ because diffraction depends on the location of every scatterer; as these locations are unknown, the diffraction effect on an A-line cannot be corrected exactly. In other words, $D(t, f)$ should not be expected to provide a deconvolution procedure in the usual sense. Conversely $D(t, f)$ should rather be considered as the deterministic quantity which describes the mean effect of diffraction. Since the context is stochastic in nature, we cannot expect to characterize the diffraction effect more precisely. This is not a restriction on the use of the diffraction filter since, in tissue characterization, random data are processed to extract certain information. The deterministic effect of diffraction is to bias these estimates. The knowledge of $D(t, f)$ allows exact unbiasing, as will be shown in the section devoted to TC algorithms. It should be remembered in dealing with random scattering media that $D(t, f)$ sums up all the information available with respect to diffraction.

5.3. Homothetic properties of the diffraction filter

The diffraction filter appears as a function of time and frequency, parameterized by the sound velocity and the geometry of the transducer. Using dimensional analysis and diffraction theory some insights into its dependence on these parameters are readily gained. Let us symbolically denote by $g$ the set of geometric parameters, homogeneous to a length, describing the transducer geometry. For a plane circular transducer $g$ reduces to the radius, for a focused transducer $g$ could be the pair (radius, focal length), etc...
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For a given type of transducer \( g \) determines the geometrical dimensions, so that \( \gamma g \) symbolically specifies a transducer identical with the one specified by \( g \), up to a scale factor \( \gamma \), i.e. a global expansion of ratio \( \gamma \). Making all the variables explicit, the diffraction transfer function at point \( r \) is written as \( H_0(g,c,r,f) \) and the diffraction filter as \( D(g,c,t,f) \). Using eq. (2.5) the behaviour of the diffraction response upon expansion of the transducer and a change in sound velocity is derived:

\[
H_0(\gamma g, c, r, f) = H_0(g, c, \frac{r}{\gamma}, \gamma f) \quad (5.3)
\]

and

\[
H_0(g, \gamma c, r, f) = H_0(g, c, r, \frac{f}{\gamma}) \quad (5.4)
\]

Introducing these relations into eq. (5.2) gives:

- Change of diffraction filter upon scaling of the transducer:

\[
D(\gamma g, c, t, f) = \gamma^6 D(g, c, t, f) \quad (5.5)
\]

- Change of diffraction filter upon a change in sound velocity

\[
D(g, \gamma c, t, f) = D(g, c, \gamma t, \frac{f}{\gamma}) \quad (5.6)
\]

5.4. Diffraction filter of the plane circular transducer

In the case of the plane circular transducer, the geometry is completely defined by the radius \( R \). A combination of the two latter equations leads to a reduced form for the diffraction filter:

\[
D(R, c, t, f) = R^6 d_1\left(\frac{ct}{R}, \frac{fR}{c}\right). \quad (5.7)
\]

Here \( d_1(x, y) \) is a dimensionless function of two dimensionless variables which completely characterizes the diffraction effect for the class of plane circular transducers. The numerical study of the diffraction filter gives an even more complete characterization in the case of the plane circular transducer. Let us consider fig. 5 where, as in fig. 4 we have plotted the quantity
$t^2 f^2 D(R, c, t, f)$, now represented by the mean of iso-amplitude lines. The striking feature of this plot is that the values of $t$ and $f$ for which the function keeps the same value are located on straight lines crossing the origin. Such a property cannot depend on particular values of $R$ and $c$ used in the numerical calculations. If this property were strictly checked, it would imply that, for each $R$ and $c$, the quantity $t^2 f^2 D(t, f)$ depends only on the ratio $f/t$. So, there exists a function $d_2$ such that:

$$t^2 f^2 D(R, c, t, f) = d_2(R, c, \frac{f}{t}).$$  \hspace{1cm} (5.8)

Putting together the two last equations implies the existence of a dimensionless function $d$ of a dimensionless variable, expressing the diffraction filter of any plane circular transducer as:

$$D(R, c, t, f) = \frac{R^6}{t^2 f^2} d\left(\frac{c^2 t}{R^2 f}\right).$$ \hspace{1cm} (5.9)

This is a powerful result since it shows that in spite of its apparent dependence on the four variables $R, c, t, f$, the diffraction filter of any plane circular transducer is determined only from the knowledge of a numerical function $d$ which can be tabulated once for all. Completion of our derivations is now simply achieved by verification of eq. 5.9 in a graphical way. We have plotted in figs 6 and 7 the values of $y = \frac{t^2 f^2}{R^6} D(R, c, t, f)$ versus the reduced variable $x = \frac{c^2 t}{R^2 f}$ for all the pairs $(t, f)$ available from our numerical evaluation.
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![Diagram](image)

Fig. 6. Values of $y = \frac{R^2 f^2}{R^6} D(R, c, t, f)$ versus the reduced variable $x = \frac{c^2 t}{R^2 f}$

of $D(t, f)$. As expected, the points gather around a curve which is then the graph $y = d(x)$ of the $d$ function.

Fig. 6 uses linear coordinates and $d$ exhibits a constant asymptote for higher values of $x$. In fig. 7 a logarithmic system coordinate is used. For the lower values of $x$, the graph is linear with a slope very close to 2, so $d$ is closely quadratic in $x$ in this region. The transition region is located around $x = 2$.

These results expressed in terms of classical diffraction terminology mean: for a circular aperture of radius $R$ at frequency $f$, a ‘Fresnel distance’ is defined as $z_F = R^2 f/c$, separating (in a fuzzy way) the ‘near field’ from the ‘far field’. The inequality $x < 2$ can then be rewritten $c t/2 < z_F$ and interpreted as: ‘at the given frequency the region of interest is the near field’. Finally, curves 4 and 5 mean: for a plane circular transducer, the limiting diffraction filter is proportional

![Diagram](image)

Fig. 7. Same as fig. 6, but log-log coordinates are used.
- to $\frac{R^2 c^4}{f^4}$ in the near field,
- to $\frac{R^6}{t^2 f^2}$ in the far field.

These two limiting forms are both separable as products of a function of time by a function of frequency. However, with the usual plane transducers the Fresnel distance is about a few centimetres; hence the $t$-$f$ domain over which the diffraction filter is not separable is precisely the region of interest and its effect is quite critical, as we will show in the next chapter.

5.5. Experimental measurement of the diffraction filter

As the diffraction filtering is a physical effect, it may be determined experimentally. We propose here a procedure by which to evaluate the diffraction effect for discrete values of time: $t_1, t_2, \ldots, t_n$ and then to relate its result to the exact diffraction filter.

The procedure has been set up as follows. Expected values of energy are found by statistical averaging. We thus used a tissue phantom made up of a rectangular piece of synthetic foam, 4 cm thick, the surface of which was far more extended than the cross-section of the ultrasonic pulsed beam. In a water tank, the transducer to be characterized is placed in front of the phantom. To measure the diffraction effect at time $t_i$, the distance $z$ from the front face of the transducer to the middle of the foam is made equal to $ct_i/2$. The transducer is operated in pulsed mode and after digitization of the received echo, a Hamming windowing centered at time $t_i$ is added in order to isolate within the response the portion of interest and the resulting power spectrum is computed. The transducer is then moved perpendicularly to its axis so that its distance to the front face of the phantom remains the same.

Another power spectrum is then computed and added to the previous one.

Fig. 8. Spatial locations of transducer and temporal locations of analysis window for diffraction filter measurements at times $t_1$ and $t_2$. 
This operation is repeated many times so that the accumulation of spectra provides the statistical averaging. The mean power spectrum is then stored on disk, concluding the measurement at the time $t_i$. Modifying the transducer distance to the phantom and the time position of the window accordingly allows the acquisition of averaged power spectra for different depths (see fig. 8). The final result is a two-dimensional array of positive values denoted $\rho_{ij}$, giving the mean value of a power spectrum for the time lag $t_i$ at the frequency $f_j$. The trick in this experiment is that the difference between two spectra is due only to the difference in propagation length or, equivalently, to the different locations of the phantom in the ultrasonic field. In other words, our experiment simulates the existence of a spatially stationary scattering medium with zero attenuation (the attenuation in water can be neglected) so that the only source of spectral variation in time is the diffraction.

Let us now examine why and how the measured spectra are directly related to the exact diffraction filter. First, as a non-attenuating medium is simulated, we can use eq. (5.1) with $a(f) = 0$, so that the diffraction filter is the only time-varying term. Of course a small attenuation effect takes place in the phantom, but this attenuation is always introduced by the same part of the phantom so that its effect is constant for all $i$ and can be included in the set of the unknown but time-invariant transfer functions. All these transfer functions are grouped in an unique term denoted $S_0(f)$ so that eq. (5.1) reads (with $a(f) = 0$): $E(\rho(t, f)) = S_0(f) D(t, f)$. Secondly, it must be recognized that the $\rho_{ij}$ are simply the sampled values of the mean spectrogram measured with the Hamming window. Due to the linearity of the expectation $E(\rho)$ the convolution relation (3.17) between $\rho(t, f)$ and the spectrogram still stands for the expected values. The Hamming window we use is $5\mu s$ long. Its spectral width, the reciprocal of the time duration, is then close to $0.2\text{MHz}$. A look at fig. 4 reveals that the variations of the measured spectra are quite smooth with respect to the extent $5\mu s \cdot 0.2\text{MHz}$, of the window in the $t-f$ plane: in the convolution (3.17) the Rihaczek distribution of the Hamming window acts in much the same way as a delta function and there is little difference between $E(\rho(t_i, f_j))$ and the mean measured spectra $\rho_{ij}$. This is why the letter $\rho$ is used here again to denote the measured samples $\rho_{ij}$ as used in sec. 4 to denote the mean energy distribution $E(\rho)$. Our conclusion is that the experiment provides the way to determine the diffraction filter up to a time-invariant frequency-dependent term. As already mentioned, this is sufficient to provide complete diffraction correction.

For the case of the planar transducer, we have proceeded to a direct comparison between the theoretical expression of the diffraction filter (numer-
cally computed) and its experimental determination (technical details are given in the next section). In such a comparison, however, the unknown time-invariant transfer function included in the experimentally determined diffraction filter has to be taken into account. This is achieved by a prior division (for all frequency values) of both time-frequency distributions by the respective values for \( t = 200 \mu s \). In this manner we arrive at distributions having the value 1 at any frequency for \( t = 200 \mu s \), which behaviour in the \( t-f \) plane can be directly compared. The experimental surface is displayed in fig. 9a and the computed one in fig. 9b. To make the comparison easier we have also multiplied both surfaces by \( f^2 \) in order to compensate for the mean temporal decrease. These two figures are now seen to be quite similar and this is the first evidence that the results actually describe the mean effect of diffraction upon the energy distribution of return signals.

5.6. Experimental results in diffraction characterization

The diffraction filters of two transducers have been experimentally estimated. The first one is a plane circular transducer manufactured at LEP with a 10 mm diameter and a central frequency of 3 MHz. The second one is a 19 mm diameter Dapco transducer focused at 90 mm with a central frequency of 3.5 MHz.

Fig. 9. Compensated (see text) diffraction filter obtained by experimental calibration (fig 9a) and by theory-based computations (fig 9b).
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Some details of the experiment are given below. The whole process of diffraction calibration has been automated using a microcomputer linked to a IEEE-488 bus. Three stepping motors control the displacement of the transducer along its axis and the 2D off-axis scanning of the foam, while a digital oscilloscope samples at 25 MHz 256 samples of the medium's response with an 8-bit resolution, transmitted to the computer which performs windowing, power spectrum calculations and averaging. For each transducer, 15 values of time corresponding to depths ranging from 3 cm to 17 cm are investigated. At each depth the transducer is moved to 200 different positions distributed over a rectangular array with steps of 2 mm.

The results are presented in figs 10a to 10e for the plane transducer and in figs 11a to 11e for the focused one. For each transducer we consider the measured running-time power, frequency and bandwidth computed according to:

\[ \bar{p}_i = \sum_j p_{ij}, \]  

(5.10)
\[
\tilde{V}_i = \frac{\sum_j f_j \rho_{ij}}{\sum_j \rho_{ij}}, \quad (5.11)
\]

and

\[
\tilde{\sigma}^2_i = \frac{\sum_j (f_j - \tilde{V}_i)^2 \rho_{ij}}{\sum_j \rho_{ij}}, \quad (5.12)
\]

Fig. 10b Energy variation versus time in six frequency bands caused by diffraction effect (plane transducer).

Fig. 11b Energy variation versus time in six frequency bands caused by diffraction effect (focused transducer).
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Fig. 10c Total energy variation versus time caused by diffraction effect (plane transducer).

Fig. 11c Total energy variation versus time caused by diffraction effect (focused transducer).

Plane transducer

Fig. 10a gives a direct plot in perspective of the values of $\rho_{ij}$, i.e. the mean power spectra as a function of time (or depth). First noticeable is the two-peak shaped power spectrum, due to the peculiar structure of the adaptation layers used experimentally with this transducer so as to ensure a large bandwidth. This feature fortunately serves our purposes since it makes it easy to observe the variation of energy for two different values of frequency. For small values of time, i.e. in the near field of the transducer, the lower frequency peak exhibits a larger amplitude than the second one. Conversely, for large values of time, i.e. in the far field of the transducer, there is an inversion and the higher frequency peak now predominates. So the effect of diffraction on the relative amplitude of these two components is to enhance the high frequency as time goes on. The same conclusion can be drawn from fig. 10b where the energy of six different frequency components is plotted versus time ($\rho_{ij}$ versus $j$ for six given values of $i$). It is again observed that
the relative loss in energy is less important when frequency increases. This is in agreement with the study of the diffraction impulse response in sec. 2, stressing that this response becomes shorter as the distance to the transducer becomes larger, thus relatively increasing the high-frequency content of the scattered signal coming from the deepest layers of tissue. The last and perhaps the most evident proof of this phenomenon is given by fig. 10d, which shows the temporal variation of the frequency centroid. The plot of $\nu_i$ versus time $t_i$ is seen to be a monotonically increasing form 2.58 MHz to 2.80 MHz. Hence the only effect of diffraction is to induce a frequency shift of 0.22 MHz: an impressive value when compared to the mean bandwidth of the signal which is about 0.7 MHz. This is a direct proof of the non-separability of the diffraction filter into a product of a function of time by a function of frequency since, in such a case, the centroid would exhibit constant values at any time. Fig. 10c shows the variation in total power: $\bar{p}_i$ versus $t_i$. As ex-
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Fig. 10e Bandwidth variation versus time caused by diffraction effect (plane transducer).

Fig. 11e Bandwidth variation versus time caused by diffraction effect (focused transducer).

Expected, a monotonic decrease in time is observed. Over the whole time range, the relative energy loss introduced by diffraction ranges up to 75%. The temporal variation of bandwidth under the influence of diffraction is shown in fig. 10e: \( \sigma_i \) versus \( t_i \). The effect is clear but of small amplitude since only 6% of variation occurs throughout the whole duration.

**Focused transducer**

Fig. 11a shows the concentration of the backscattered energy around focal time. The geometrical focal length determined by the curvature of the transducer face corresponds (with \( c = 1500 \text{ m} \cdot \text{s}^{-1} \)) to a 120µs focal time. Drawing in fig. 11b the time variation of energy at 6 different frequencies unexpectedly reveals that the maxima of backscattered energy are rather located around 100µs. This can be explained by recalling that the diffraction transfer function is obtained by summation of Green functions originating from the transducer face (eq. 2.10). The maximum phase coherence of these
functions is at the focus, but their amplitudes are affected by a propagation loss term in $r^{-1}$ which makes the maximum of backscattered energy appear before the geometrical focal time. The total energy variation is displayed in fig. 11c. The frequency centroid $\tilde{f}_i$ is plotted versus time in fig. 11d with noticeable variations from 3.83 MHz to 3.54 MHz, giving throughout the duration a peak-to-peak deviation of 0.29 MHz. The maximum value observed around focal time is easily understood by recalling that the diffraction impulse response is of minimum duration on the focal plane, corresponding to a larger high-frequency content. Temporal evolution of the bandwidth, $\tilde{\sigma}_i$ is given in fig. 11e. The peak-to-peak relative variation throughout the duration is 17%.

6. Algorithms for attenuation estimation

6.1. General considerations

Tissue characterization is not an easy task because biological soft tissues are complex in nature and many physical phenomena contribute to the finally observed signal. The simultaneous evaluation at any point in the tissue of the values of sound velocity, scattering response, scatterer density, etc., seems out of reach. That is why most of the published algorithms for attenuation estimation assume that the portion of tissue to be examined is homogeneous. This is a very simplifying assumption since it allows one to impute any change in the signal features only to the propagation effect, which is then the only source of non-stationarity in the ultrasound signal. Owing to a lack of a sufficient modelling or experimental evidence, however, the diffraction phenomenon is often ignored so that the mean power spectrum at time $t$, denoted here $P(t, f)$, usually takes into account only the attenuation.

$$P(t, f) = S_o(f) \exp(-2a(f)c t).$$  \hspace{1cm} (6.1)

In the foregoing, we have shown for scattering media how the diffraction filter can simply describe the actual influence of propagation phenomena on energy distribution, thus providing the basic formula for attenuation estimation:

$$P(t, f) = S_o(f) \exp(-2a(f)c t) D(t, f).$$  \hspace{1cm} (6.2)

As the diffraction term appears multiplicatively in eq. (6.2) with respect to eq. (6.1) and because all published methods are based (explicitly or not) on
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eq. (6.1), defining a diffraction correction procedure to unbias them is not difficult (at least conceptually). This will be detailed in the following sections, where we examine three different kinds of methods.

A last point has to be clarified. Basic equation (6.2) have been derived for the mean value of the Rihaczek distribution. On the other hand, the algorithms discussed use more classical spectral analysis. In fact, their spectral estimates are always obtained by windowing followed by Fourier transformation. This procedure is equivalent to the use of the spectrogram, which is shown to be related to the Rihaczek distribution by a double convolution in both time and frequency with the energy distribution of the analysis window (3.3.2). As \( P(t, f) \) is smoothly varying (partly because it is a mean spectrum) the window acts in this 2D convolution as a delta function so that eq. (6.2) is valid for the Rihaczek distribution and for the spectrogram as well. The ‘window bias’ in the reviewed algorithms will therefore be neglected and we shall focus on the ‘diffraction bias’.

6.2. Narrow-band algorithm

In this method \(^7\), the decay of echographic signal power in different frequency bands is used as an indicator of the attenuation effect. If the diffraction effect is at first ignored, relation (6.2) indicates that the power of the signal passed through a narrow-band filter centered around the frequency \( f_i \) decays according to \( \exp(-2a(f_i)ct) \).

Hence the time derivative of the logarithm of the band-passed signal power yields the value of attenuation at the given frequency. In practice a linear regression in time is performed over the time range of interest and gives an estimate \( a(f_i) \) of the attenuation at that frequency. This can be repeated at different frequencies in the signal spectrum, and if a linear attenuation law is assumed according to \( a(f_i) = \beta f_i \), then a linear regression carried out on measured values of \( a(f_i) \) yields an estimation of the attenuation coefficient \( \beta \). In practice the signal to be analysed at a given frequency \( f_i \) is heterodyned at that frequency and passed through a constant low-pass filter whose impulse response duration determines the time resolution of the method.

The diffraction correction for this method is quite evident. From eq. (6.2) the mean value of the signal power in a narrow frequency band centered around \( f_i \) actually varies as \( \exp(-2a(f_i)ct)D(t,f_i) \). Hence, before performing the linear regression in time on the output of the bandpass filter, this output has to be divided by \( D(t,f_i) \) so that only the attenuation effect remains and is measured. A typical amplitude of such a correction is given in sec. 5 with examples for both a plane and a focused transducer. In the case of the focused transducer, which naturally introduces strong variations around
the focal zone, the corrected bias can reach a value of 45%. This diffraction bias would still exist if the diffraction filter were not separable.

6.3. Spectral difference algorithm

Whereas different spectral ‘slices’ are processed independently in the frequency band algorithm, the spectral difference algorithm\(^8,9\) deals with the whole spectrum. The method is based on the comparison of two power spectra measured at two different times \(t_1\) and \(t_2\). According to formula (6.1), i.e. neglecting diffraction and window effects, the log ratio of these two spectra equals, on average, the quantity \(-2a(f)c(t_2-t_1)\). Seeking a linear attenuation, the coefficient \(\beta\) is obtained after a linear regression in frequency of the log ratio throughout the useful bandwidth. In this method, the time-invariant unknown (or uncalibrated) transfer functions, regrouped in the term \(S_o(f)\), disappear because a ratio of power spectra is considered.

Nonetheless the diffraction effect being not time-invariant still plagues this method and must be taken into account. The diffraction filter being known, this is easily achieved and formula (6.2) again gives the key to diffraction correction since it yields the actual log ratio of power spectra:

\[-2a(f)c(t_2-t_1) + \ln \frac{D(t_2,f)}{D(t_1,f)}\]

If the diffraction filter were separable into the product of a function of time by a function of frequency, then the function of frequency would be cancelled in this spectral difference while the difference of time functions would only add an offset to the log ratio. So, as far as only the slope of the log spectral ratio is considered, a separable diffraction filter would have no influence on the attenuation coefficient estimates. However, as already pointed out in sec. 5, this is not generally the case: the diffraction filter is not separable and the diffraction effect still has to be corrected by subtracting the diffraction contribution \(\ln (D(t_2,f)) - \ln (D(t_1,f))\) to the log spectra ratio.

6.4. Frequency shift algorithm

This method is based on measurement of the frequency centroid downshift in time. Its derivation is not as straightforward as for the two previous algorithms and is based explicitly on the hypotheses of linear attenuation. Let us start with eq. (6.1) as the expression for mean distribution of energy in time and frequency. As defined in sec. 3, the running-time spectral moment of order \(n\) is in this case given by:

\[\overline{m}_n(t) = \int S_o(f) \exp(-2 f c t) f^n df.\]  

(6.3)
The time derivative of successive moments then verifies:

\[
\frac{d}{dt} \bar{m}_n(t) = -2\beta c \bar{m}_{n+1}(t).
\]  

(6.4)

Therefore

\[
\frac{d}{dt} \frac{\bar{m}_1}{m_0} = -2\beta c \left\{ \frac{m_2}{m_0} - \left( \frac{\bar{m}_1}{m_0} \right)^2 \right\}.
\]

(6.5)

We can recognize \( \frac{\bar{m}_1}{m_0} \) as the instantaneous frequency \( \dot{\nu}(t) \) for the mean energy distribution and in the right-hand term the instantaneous squared bandwidth calculated for the same distribution, so that we get the basic equation for the frequency shift algorithm:

\[
\beta c = \frac{-1}{2\sigma^2} \frac{d\dot{\nu}}{dt}.
\]

(6.6)

This shows how the attenuation coefficient can be derived from the running-time measurements of the instantaneous frequency and bandwidth of the echographic signal. It is usually reported in literature that this method relies on the assumption of a Gaussian-shaped power spectrum; nonetheless it is shown here that the basic relation (6.6) can be derived whatever the shape of the spectrum is, provided that the bandwidth be measured simultaneously with the frequency decay. Of course, diffraction correction remains necessary and is simply achieved by dividing the measured running-time spectra by the diffraction filter prior to the computations of running-time frequency moments. As a final remark before discussing the implementation of the method and describing the experiments, we want to stress that diffraction correction would not be necessary if the diffraction filter were separable into a product of a function of time by a function of frequency. This is because the function of frequency could be merged with \( S_0(f) \) while the function of time would have no influence over spectral centroid and bandwidth since they are normalized moments.

6.5. Implementation of a diffraction-corrected attenuation estimator

Special attention has been devoted to the spectral shift method, and a complete experimental proof of its efficiency when combined with a cor-
Fig. 12. a) raw energy distribution in time and frequency; b) diffraction filter for the corresponding time-frequency range; c) diffraction-corrected energy distribution in time and frequency.

rectly designed diffraction correction preprocessing has now been given. For this purpose we used the focused transducer and its measured diffraction filter as described in sec. 5.4.

While the diffraction calibration was carried out on a foam phantom, the attenuation measurements were performed on a tissue phantom of a different kind, made up of a gel including small scattering targets*). An attenuation coefficient of 0.365 dB/cm·MHz., was measured by a transmission technique. This phantom has a cylindrical shape with a diameter of 7.5 cm

* Tissue mimicking phantoms were provided by Dr. J.A. Zagzebski (Department of Medical Physics, University of Wisconsin, USA)
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and length 5 cm, allowing a double test: in a first set of data the front face of the phantom is located 50 mm away from the transducer (before focus) while in the second set the phantom is at 130 mm (behind focus). The electronic and mechanical set-ups are similar to the ones described in sec. 5.5 and allow the recording of 30 A-lines for each set of data.

The averaged running-time power spectra are displayed in figs 12a before (in front of) focus and 13a after (behind) focus. Their differences are striking and are due to the vicinity of the focal point. Figures 12b and 13b show plots of the portions of the diffraction filter corresponding to the same domain in time and frequency (since sampling in time and space were not set
Fig. 14. Phantom located before the transducer focal point; a) frequency (solid line) and squared bandwidth (dotted line) of the plot 12a (before diffraction correction); b) frequency (solid line) and squared bandwidth (dotted line) of the plot 12c (after diffraction correction).

equal for diffraction calibration and for attenuation estimation, interpolation of the diffraction filter is needed; a linear interpolation seems sufficient. For the first set, the focal point corresponds to the end of the data record so that the diffraction effect in this zone is to increase the signal level (fig 12b), acting against the attenuation effect in such a way that the power in fig 12a does not seem to decrease. Conversely, the second set has been recorded beyond the focal point so that the diffraction now acts in the same direction as the attenuation (fig 13b) and the power in fig 13a exhibits a strong decrease, stronger than if the attenuation had been the unique cause of temporal variation.

Results of diffraction correction are shown in fig 12c and 13c which give simple plots the running-time spectra of figs 12a and 13a after division by
the corresponding diffraction filters of figs 12b and 13b respectively. Although some noise is present, the running-time spectra now exhibit the same behaviour. It is worth noticing that if diffraction calibration had been carried out on the same phantom as the one to be characterized without modifying the acquisition system, then figs 12c and 13c would just show surfaces in \( \exp(-2\beta c ft) \) since only the attenuation effect would have differentiated them. (This is because we do not use the theoretical diffraction filter but its experimentally determined version, which still includes the time-invariant transfer functions due to excitation, transduction etc. and which would otherwise exactly cancel in the dividing operation of the diffraction correction procedure).

It remains to compute the running-time spectral moments by integration...
from 2MHz to 6MHz, still confronting before-after focus and with-without
diffraction correction. The four following figures show plots versus time of
the frequency centroid and, as a dotted line, the squared bandwidth. In each
case, the centroid slope is estimated by linear regression, while the squared
bandwidth is considered as constant in the range and its value is estimated
by a simple averaging. The estimated value of $\beta$ is reported in each figure.
Let us first compare the estimates before and after focus without diffraction
correction. In the before-focus situation (fig. 14a) the frequency downshift
due to attenuation is masked by diffraction, the impulse response becomes
sharper as time tends towards 120 $\mu$s (focal time). As a result, an almost
zero attenuation coefficient is estimated. On the contrary, after focus (fig.
15a) diffraction and attenuation act in the same way and a very strong fre-
quency downshift is experienced. As a result, the attenuation coefficient is
overestimated by almost a factor of 2. Let us now compare the estimates after
diffraction correction. The measured bandwidth has artificially increased be-
cause division of the spectra by the experimental diffraction filter naturally
introduces flattening and also allows a wider range of variation to the fre-
quency centroid. Fig 14b represents the ‘before focus’ configuration where
the centroid has recovered a decreasing behaviour. The estimated value for
$\beta$ is now 0.378 which differs from the actual value by only 4 %. In fig 15b
the ‘after focus’ case is reported with an estimated value of 0.364, almost
equal to the actual value. These results are (unfortunately!) too good, be-
cause they do not reflect the actual variance of the algorithm for which a
complete study should be provided. However, they high-light the diffraction
bias problem to which answers are clearly given by these experiments: the
diffraction bias has to be corrected and the proposed method gives very sat-
isfactory results.

Conclusion and perspectives

The last section devoted to algorithms provides an unifying description of
different methods aimed at tissue characterization. In this description the
physical phenomena involved in the formation of the return signal are identi-
fied and their effects on energy distribution are now clear. The experimen-
tal results presented here show that the proposed approach actually pro-
vides a correct formalization for tissue characterization tasks. Special
attention has been devoted to diffraction since its effect has usually been ig-
nored or neglected. Although convincing arguments for the necessity of dif-
fraction correction have been given, we are aware that its calibration for each
transducer to be used requires a very substantial amount of experimental
work. Of course, in the case where the diffraction impulse response of the
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transducer can be computed, the theoretical expression of the diffraction filter is easily obtained and used for correction. However this still represents a heavy computational task. That is why further work is needed to give the diffraction filter an analytic expression valid for any plane or focused transducer. It would then be possible to design a simple time-varying filter, parameterized by radius and focal distance, correcting the mean diffraction effect and operating directly in the time domain.

Since we have used a complete three-dimensional model, we believe that our results are fairly general and provide a sound theoretical basis for obtaining unbiased estimates of the tissue parameters. For medical systems this is of the utmost importance in experimental clinical validation and ultimately in routine medical examinations.

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