THEORY OF A THIN CONDUCTING STRIP WITH AN OVERHEAD RETURN WIRE

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Abstract

We determine the primary transmission parameters (capacitance and impedance per unit length) of a line consisting of a thin horizontal strip of finite or infinite width, and of a thin return wire situated above the centre of the strip. In the analysis of the lateral skin- and proximity-effects, the cross-section of the strip of finite width is assumed to be a thin ellipse.

Keywords: Bessel functions, capacitance, current distribution, electric impedance, electrostatics, high-frequency asymptotics, skin effect, transmission lines.

1. Introduction

Our main purpose is to obtain analytical expressions for the primary transmission parameters per unit length (capacitance \(C\), and impedance \(Z = R + i\omega L\)) of a line consisting of a thin horizontal strip of finite width \(2a\) and of a thin return wire situated at a distance \(c\) above the centre of the strip. In the Cartesian coordinates \(x, y, z\), the transmission line is infinite in the \(z\)-direction, the thin strip has a centre line \(|x| \leq a, y = 0\) and the return wire is a cylinder of small radius \(r\) centred at \(x = 0, y = c\); the cross-section of the line is thus as shown in fig. 1. The thin strip of thickness \(h\) is assumed to extend symmetrically (from \(y = -h/2\) to \(y = h/2\)) below and above its centre line, but \(h\) does not need to be constant, and various profile laws \(h(x)\) will in

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fact be considered. The strip has conductivity $\sigma$ and permeability $\mu$, but the corresponding constants of the return wire are not specified, because the determination of its internal impedance is a separate well-known problem.

The computation of the primary line parameters is done according to the accepted methodology of engineering transmission line theory. The capacitance per unit length is computed by solving the electrostatic problem where the strip carries a charge $q$ per unit length and the return wire a charge $-q$. The impedance per unit length is computed by treating the go-conductor (i.e. the strip) as carrying an alternating current $I e^{j\omega t}$ which returns by the return wire and by using the quasi-stationary approximation (Maxwell's equations with the displacement current neglected) in the computation of the electromagnetic fields. In the engineering theory, the secondary line parameters (propagation constant and characteristic impedance) are then deduced from the primary line parameters by the telegraph equations. For perfect conductors in the TEM mode, it is well known that this engineering theory is equivalent to the rigorous theory based on the full Maxwell equations\(^1\). For imperfect, but good conductors, the engineering theory is known to be an excellent approximation to the rigorous theory\(^2\) provided the distance between the go and return conductors is small compared with the wavelength.

The electrostatic problem is treated in Sec. 2. For a strip of infinite width ($a = \infty$), the problem is elementary by the method of images. To our best knowledge, the strip of finite width has not been treated in the literature, except for a brief allusion by Durand\(^3\).

The rest of the paper deals with the electromagnetic problem for a thin strip, which means that its thickness $h$ is small with respect to the skin-depth

$$\delta = (2/\omega \mu \sigma)^{1/2},$$  \hspace{1cm} (1.1)

i.e.

$$h \ll \delta.$$  \hspace{1cm} (1.2)
Consequently, the skin-effect in the strip is purely lateral: the current density (A m\(^{-2}\)) is uniform along the thickness coordinate (y in fig. 1) and the only problem is to determine the lateral distribution of \(j(x)\), the linear current density (A m\(^{-1}\)), along the width coordinate (x in fig. 1).

The lateral problem with a return wire at large distance has been treated in several papers \(^4,5\). The foundations of the lateral problem with a return at finite distance are handled in Sec. 3. The asymptotic lateral problem at high frequencies is solved in Sec. 4. The general lateral problem for the infinitely wide strip \((a = \infty)\) of uniform thickness is solved in Sec. 5. The treatment of the general problem for finite \(a\) is continued in Sec. 6.

In Sec. 7, we consider a finite width \(2a\) but assume that the strip cross-section is a very thin ellipse of semi-axes \(a\) and \(b\), with \(b \ll a\), because the thin rectangular cross-section is almost intractable \(^5\). In the quoted reference, the return conductor was a cylinder of large radius surrounding the strip; in the present paper, the return wire is also at large distance for \(c = \infty\), and we recover our previous results in that case. For finite \(c\), however, the return wire adds a lateral proximity effect to the strip's own lateral skin-effect. Low-frequency approximations and high-frequency asymptotics are discussed in Secs 8 and 9, respectively. Finally, it is shown in the Appendix that the results of Sec. 5 can be deduced from those of Sec. 7 for \(a \to \infty\).

2. The electrostatic problem

The present section deals with the two-dimensional electrostatic problem of the transmission line described in Sec. 1 (fig. 1). The line is infinite along Oz and consists of a strip \((|x| \leq a, y = 0)\) carrying a charge \(q\) per unit length, and of a thin return wire \((x = 0, y = c)\) carrying a charge \(-q\). We determine the potential, the capacitance per unit length and the charge density on the strip. In the electrostatic problem, all conductors are perfect, and the thickness of the strip (which may even be a function of \(x\)) is irrelevant, provided it is small with respect to its width \(2a\).

The potential \(V\) satisfies the Laplace equation in \(x, y\), is regular everywhere except for a logarithmic singularity at the centre of the return wire created by the charge \(-q\), and is constant on the strip. The solution of this classical potential problem is known to be unique if one requires the potential to vanish at large distance \((x^2 + y^2 \to \infty)\).

In the two-dimensional geometry of fig. 1, the Cartesian coordinates \(x, y\) are combined into the complex variable \(z = x + iy\) (not to be confused with the space \(z\)-coordinate, no longer used). The Joukowski transformation

\[
z = \frac{1}{2}a(\zeta + \zeta^{-1}); \quad \zeta = \xi + i\eta = \rho \, e^{i\theta} \quad (-\pi \leq \theta \leq \pi),
\]  

\((2.1)\)
with the restriction $\rho \geq 1$, defines a one-to-one mapping of the $z$-plane onto the outside of the unit circle of the $\zeta$-plane; the strip $|x| \leq a$, $y = 0$ becomes the unit circle $\rho = 1$, the upper (lower) side of the strip corresponding to the upper (lower) half-circle $0 \leq \theta \leq \pi$ ($-\pi \leq \theta \leq 0$). The position $z = ic$ of the return wire transforms into

$$\zeta = i e^\phi,$$

(2.2)

where

$$\frac{c}{a} = \sinh \phi.$$  

(2.3)

The mapping (2.1) transforms the original problem into a potential problem in $\zeta \geq 1$, which is readily solved by the method of images: in addition to the line charge $-q$ at (2.2), we introduce the line charge $+q$ at the image point $\zeta = i e^{-\phi}$ inside the unit circle. The combined potential

$$V = \frac{q}{2\pi \varepsilon} \log \left| \frac{\zeta - i e^\phi}{\zeta - i e^{-\phi}} \right|,$$

(2.4)

where $\varepsilon$ is the dielectric constant, vanishes at large distance ($|z| \to \infty$, $|\zeta| \to \infty$) and takes a constant value on the unit circle $|\zeta| = 1$. Indeed, by setting $\zeta = e^{i\theta}$, one has

$$\left| \frac{\zeta - i e^\phi}{\zeta - i e^{-\phi}} \right| = \left| \frac{e^{i\theta} - i e^\phi}{e^{i\theta} - i e^{-\phi}} \right| = \left| \frac{-i e^\phi 1 + i e^{i\theta - \phi}}{-i e^{i\theta} 1 - i e^{-i\theta - \phi}} \right| = e^\phi,$$

(2.5)

so that the potential (2.4) takes the constant value

$$V_+ = \frac{q\phi}{2\pi \varepsilon}$$

(2.6)

on the strip, as required.

The potential $V_-$ on the surface of the thin wire of radius $r$ can be computed at an arbitrary point of its surface, say at $x = 0$, $y = c + r$, hence at $z = i(c + r)$. For small $r$, the corresponding point in the $\zeta$-plane is near (2.2), say at $\zeta = i(e^\phi + \lambda)$ with small $\lambda$. From (2.1) one deduces

$$c + r = \frac{1}{2a} \left( e^\phi + \lambda - \frac{1}{e^\phi + \lambda} \right).$$
To first order in $\lambda$, this yields, using (2.3)

$$\lambda = \frac{2r}{a(1 + e^{-2\phi})} = \frac{r e^{\phi}}{a \cosh \phi}. \quad (2.7)$$

To the same order, the resulting value of the left-hand side of (2.5) is

$$\left| \frac{\zeta - i e^{\phi}}{\zeta - i e^{-\phi}} \right| = \left| \frac{i \lambda}{i(e^{\phi} - e^{-\phi})} \right| = \frac{\lambda}{2 \sinh \phi},$$

hence, by (2.7)

$$\frac{r e^{\phi}}{2a \sinh \phi \cosh \phi} = \frac{r e^{\phi}}{a \sinh(2\phi)},$$

so that we have

$$V_- = \frac{q}{2\pi \varepsilon} \left\{ \log \left[ \frac{r}{a \sinh(2\phi)} \right] + \phi \right\}. \quad (2.8)$$

The capacitance is

$$C = \frac{q}{V_+ - V_-} = \frac{2\pi \varepsilon}{\log[(a/r) \sinh(2\phi)]}. \quad (2.9)$$

Using (2.3) this becomes

$$C = \frac{2\pi \varepsilon}{\log[(2c/r) \cosh \phi]} = \frac{2\pi \varepsilon}{\log(2c/r) + \frac{1}{2} \log(1 + c^2/a^2)}. \quad (2.10)$$

The inverse of the transformation (2.1) is the solution for $|\zeta| \geqslant 1$ of a quadratic equation in $\zeta$, giving

$$\zeta = \frac{z}{a} + \left[ \left( \frac{z}{a} \right)^2 - 1 \right]^{1/2}. \quad (2.11)$$

For large $a$ and to first order in $z/a$, the result (2.11) becomes

$$\zeta = i + z/a, \quad \text{Im} \, z > 0. \quad (2.12)$$
Also for large $a$ one has $\phi = c/a$ from (2.3) and $e^{\pm \phi} = 1 \pm c/a$. Hence the potential (2.4) becomes

$$V = \frac{q}{2\pi \varepsilon} \log \left| \frac{z - ic}{z + ic} \right| = \frac{q}{4\pi \varepsilon} \log \left[ \frac{x^2 + (y - c)^2}{x^2 + (y + c)^2} \right], \quad (y \geq 0) \quad (2.13)$$

which is the elementary result obtained by the method of images for the infinitely wide strip ($a = \infty$). Also the expression (2.10) then becomes

$$C = \frac{2\pi \varepsilon}{\log(2c/r)}, \quad (2.14)$$

which can be derived by the method of images by considering a pair of parallel wires of radii $r$ separated by a distance $2c$. The known capacitance of such a pair is one-half of (2.14) because it is equivalent to two capacitances (2.14) (between each wire and the symmetry plane) connected in series, the symmetry plane at zero potential being ignored.

The charge density on the upper side of the strip is

$$\sigma_+ = -\varepsilon \left. \frac{\partial V}{\partial y} \right|_{y=0+}. \quad (2.15)$$

Now, the potential (2.4) is the real part of the analytic function

$$\Omega(\zeta) = \frac{q}{2\pi \varepsilon} \log \left[ \frac{\zeta - i e^\phi}{\zeta - i e^{-\phi}} \right] = V + iW, \quad (2.16)$$

so that $\partial V/\partial y$ can be expressed in terms of the derivative of $\Omega(\zeta)$ as

$$\frac{\partial V}{\partial y} = -\text{Im} \frac{d\Omega}{dz} = -\text{Im} \left( \frac{d\Omega}{d\zeta} \frac{d\zeta}{dz} \right) = -\frac{2q \sinh \phi}{\pi \varepsilon a} \text{Re} \left[ \frac{\zeta^2}{(\zeta^2 - 1)(\zeta^2 - 2i \cosh \phi - 1)} \right]. \quad (2.17)$$

As mentioned in the text following (2.1), it is recalled that, in terms of the parameter $\theta$, the upper (lower) side of the strip is described by $x = a \cos \theta$, $y = 0+$ ($y = 0-$) where $0 \leq \theta \leq \pi$ ($-\pi \leq \theta \leq 0$). Then, from (2.15) and
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(2.17), the charge density on the upper side is found to be

$$\sigma_+(\theta) = \frac{2q \sinh \phi}{\pi a} \text{Re} \left[ \frac{1}{(e^{i\theta} - e^{-i\theta})(e^{i\theta} - 2i \cosh \phi - e^{-i\theta})} \right]$$

$$= \frac{q}{2\pi a} \frac{\sinh \phi}{\sin \theta (\cosh \phi - \sin \theta)}, \quad (2.18)$$

and this gives $\sigma_+(x)$ for $x = a \cos \theta$, $0 \leq \theta \leq \pi$. Similarly the density on the lower side is

$$\sigma_-(\theta) = \varepsilon \frac{\partial V}{\partial y} \bigg|_{y=0} = \frac{q}{2\pi a} \frac{\sinh \phi}{\sin \theta (\cosh \phi - \sin \theta)}, \quad (2.19)$$

where $x = a \cos \theta$, $-\pi \leq \theta \leq 0$. By setting

$$\sin \theta = \pm (1 - x^2/a^2)^{1/2}, \quad (2.20)$$

where the upper sign is to be taken in (2.18) but the lower sign in (2.19), these equations become, using (2.3),

$$\sigma_\pm(x) = \frac{q}{2\pi a} \frac{c/a}{(1 - x^2/a^2)^{1/2} [(1 + c^2/a^2)^{1/2} \mp (1 - x^2/a^2)^{1/2}]}$$

$$= \frac{qc}{2\pi a^2} \frac{(1 + c^2/a^2)^{1/2} \pm (1 - x^2/a^2)^{1/2}}{(1 - x^2/a^2)^{1/2}(c^2 + x^2)/a^2}$$

$$= \frac{qc}{2\pi} \frac{[(a^2 + c^2)/(a^2 - x^2)]^{1/2} \pm 1}{c^2 + x^2}. \quad (2.21)$$

We now evaluate the total charges

$$q_\pm = \int_{-a}^{a} \sigma_\pm(x) \, dx \quad (2.22)$$

on either side of the strip. The integration of (2.21) is elementary and we find

$$q_\pm = \frac{1}{2} q \left[ 1 \pm \frac{2}{\pi} \arctan \left( \frac{a}{c} \right) \right]. \quad (2.23)$$
As expected, we have

\[ q_+ + q_- = q \] (2.24)

but eq. (2.23) shows that the proximity effect of the return wire attracts an excess charge on the upper side of the strip.

The distribution of \( \sigma_+(x) \) is best discussed by noting that the derivative with respect to \( \theta \) of the denominator of eq. (2.18) vanishes for \( \theta = \pi/2 \), i.e. at the centre \( x = 0 \) of the strip, and at the solutions in \( \theta \) of the equation

\[ 2 \sin \theta = \cosh \phi \]

For \( c/a > \sqrt{3} \), the value (2.25) is imaginary and the function (2.18) has a single extremum, a minimum

\[ \sigma_+ = \frac{q}{2\pi c} \left[ (1 + c^2/a^2)^{1/2} + 1 \right] \] (2.26)

at \( x = 0 \). For \( c/a < \sqrt{3} \), the function (2.18) has a maximum given by (2.26) for \( x = 0 \) and two minima

\[ \sigma_+ = \frac{2q}{\pi c(1 + a^2/c^2)} \] (2.27)

at (2.25). By contrast, \( \sigma_- \) has a single minimum

\[ \sigma_- = \frac{q}{2\pi c} \left[ (1 + c^2/a^2)^{1/2} - 1 \right] \] (2.28)

at \( x = 0 \).

In (2.21), both \( \sigma_+ \) and \( \sigma_- \) become infinite at the ends \( x = \pm a \) of the strip. These singularities disappear for an infinitely wide strip \( (a = \infty) \) where one has

\[ \sigma_+ = \frac{qc}{\pi(c^2 + x^2)}; \quad \sigma_- = 0 \] (2.29)
whereas eq. (2.23) yields

\[ q_+ = q; \quad q_- = 0 \]  

(2.30)

which are the expected results by the method of images.

For a return at large distance \( (c = \infty, \phi = \infty) \), eq. (2.23) yields \( q_+ = q_- = q/2 \) whereas eq. (2.21) gives

\[ \sigma_+ = \sigma_- = \frac{q}{2\pi(a^2 - x^2)^{1/2}}. \]  

(2.31)

3. Foundations of the quasi-stationary electromagnetic problem

We now consider the transmission line (infinite along Oz) whose cross-section is represented in fig. 1, when the strip \( (|x| \leq a, y = 0) \) carries an alternating current \( I e^{i\omega t} \) which returns by the thin wire of radius \( r \) centred at \( x = 0, y = c \). In the quasi-stationary approximation, the displacement current is neglected everywhere in Maxwell’s equations, so that propagation is also neglected and the current \( I \) is independent of \( z \). As stated in Sec. 1, the conductivity of the return wire need not be specified, whereas the strip has conductivity \( \sigma \) and a thickness \( h = h(x) \), small with respect to the skin-depth. Consequently, the real strip can be replaced by an infinitely thin strip, of surface conductivity \( \sigma h \), located at \( |x| \leq a, y = 0 \). The time dependence factor \( e^{i\omega t} \) is suppressed in the sequel.

In the quasi-stationary theory, the magnetic field \( H \) and the electric field \( E \) are classically expressed in terms of the vector potential \( A \) and the scalar potential \( V \), by

\[ H = \frac{1}{\mu} \text{curl} A, \]  

(3.1)

\[ E = -i\omega A - \nabla V. \]  

(3.2)

By imposing the gauge condition \( \text{div} A = 0 \), the potentials \( A \) and \( V \) must satisfy the Laplace equation outside the strip and the return wire. Since \( A \) is created by the currents \( \pm I \) in the conductors, the only non-zero component of \( A \) is \( A_z \), to be denoted by \( \Psi \), which must be proportional to \( I \) and independent of \( z \). The function \( \Psi(x, y) \) is harmonic and regular everywhere, except on the strip and except for a logarithmic singularity at the centre of the return wire.
where its singular part is

$$\frac{\mu I}{2\pi} \log[(x^2 + (y - c)^2)^{1/2}]. \quad (3.3)$$

Moreover, since \( \Psi \) is only defined within a constant, one may require it to vanish at large distance \((x^2 + y^2 \to \infty)\).

The linear current density on the strip points in the \( z \)-direction and is found as the jump of the tangential component of \( H \) across the strip, so

$$j(x) = H_x|_{y=0} - H_x|_{y=0^+} = \frac{1}{\mu} \left( \frac{\partial \Psi}{\partial y}|_{y=0^+} - \frac{\partial \Psi}{\partial y}|_{y=0^-} \right) \quad (3.4)$$

by (3.1). Consequently, \( \partial \Psi/\partial y \) is discontinuous across the strip. By contrast, \( \Psi \) itself is continuous because so is the normal component \( H_z = -\frac{1}{\mu} \partial \Psi/\partial x \). Moreover, the condition

$$\int_{-a}^{a} j(x) \, dx = I \quad (3.5)$$

is an automatic consequence of (3.4) and of \( \Psi \) being harmonic outside the strip except for the singularity (3.3).

Since \( E_x = 0 \) on the strip (because there is no current in the \( x \)-direction) and since \( A_x = 0 \), eq. (3.2) yields \( \partial V/\partial x|_{y=0} = 0 \). Consequently, \( V|_{y=0} \) is a function of \( z \) alone. Since by Ohm’s law one has

$$E_z = \frac{j}{\sigma h} \quad (3.6)$$

on the strip, the \( z \)-component of (3.2) yields

$$-\frac{\partial V}{\partial z}|_{y=0} = i\omega \Psi|_{y=0^+} + \frac{j}{\sigma h} \quad (3.7)$$

In (3.7), the right-hand side is a function of \( x \) alone, whereas we just established that \( V|_{y=0} \), hence also \( \partial V/\partial z|_{y=0} \), was a function of \( z \) alone. Consequently, both sides of (3.7) are constants. The constant value

$$U_+ = -\frac{\partial V}{\partial z}|_{y=0} \quad (3.8)$$
is the voltage drop per unit length on the strip, and we rewrite (3.7) as

\[ U_+ = i\omega \Psi \bigg|_{y=0} + \frac{j}{\sigma h} \]  

(3.9)

Eliminating \( j \) between (3.4) and (3.9), we obtain

\[ U_+ = i\omega \Psi \bigg|_{y=0} + \frac{1}{\mu \sigma h} \left( \frac{\partial \Psi}{\partial y} \bigg|_{y=0^-} - \frac{\partial \Psi}{\partial y} \bigg|_{y=0^+} \right). \]

(3.10)

The constancy of (3.10) on the strip imposes a boundary condition on \( \Psi(x, y) \) in addition to the other specifications on \( \Psi \) described in the text surrounding (3.3). Notice that the constant \( U_+ \) is as yet unknown and is to be determined as part of the solution of the boundary value problem for \( \Psi \). We briefly discuss the uniqueness of the solution for \( \Psi \) and \( U_+ \). To that end we suppose the boundary value problem has two solutions \( \{\Psi^{(1)}, U^{(1)}_+\} \) and \( \{\Psi^{(2)}, U^{(2)}_+\} \). If we introduce the difference

\[ W(x, y) = \Psi^{(1)}(x, y) - \Psi^{(2)}(x, y) - \frac{1}{i\omega} (U^{(1)}_+ - U^{(2)}_+), \]

(3.11)

then \( W \) is harmonic and regular outside the strip, while the condition (3.10) turns into the homogeneous boundary condition

\[ \frac{\partial W}{\partial y} \bigg|_{y=0^-} - \frac{\partial W}{\partial y} \bigg|_{y=0^+} + i\omega \mu \sigma h W \bigg|_{y=0} = 0 \]

(3.12)

on the strip. We now apply Green's first identity to \( W \) and its complex conjugate \( W^* \) over the \( (x, y) \)-plane (\( \mathbb{R}^2 \)) outside the strip. As a result we obtain

\[ \iint_{\mathbb{R}^2} \text{grad } W \, dx \, dy = \int_{-a}^{a} W^*(x, 0) \left[ \frac{\partial W}{\partial y} (x, 0^-) - \frac{\partial W}{\partial y} (x, 0^+) \right] dx \]

\[ = -i\omega \mu \sigma \int_{-a}^{a} h(x)|W(x, 0)|^2 \, dx \]

(3.13)

by use of (3.12). Obviously this identity can only hold if both sides of (3.13) vanish, which implies that \( \text{grad } W = 0 \) in \( \mathbb{R}^2 \) and \( W = 0 \) on the strip, hence \( W = 0 \) in \( \mathbb{R}^2 \). By setting \( W = 0 \) in (3.11) and by recalling that \( \Psi^{(1)}, \Psi^{(2)} \) vanish at large distance, it is readily seen that \( \Psi^{(1)} = \Psi^{(2)} \) and \( U^{(1)}_+ = U^{(2)}_+ \).
This proves the uniqueness of the solution for \( \Psi \) and \( U_+ \).

Finally, we rewrite (3.9) as

\[
j(x) = \sigma h(U_+ - i \omega \Psi|_{y=0})
\]

which allows us to compute the linear current density more simply than by (3.4).

On the surface of the thin return wire \( E \) is constant and its value is \( E_z = -Z_i I \), where \( Z_i \) is the internal skin-effect impedance per unit length, computed as if the thin wire existed alone (the expression for \( Z_i \) is well known, involving the ratio \( J_0/J_1 \) of Bessel functions). Calling \( U_\) the voltage drop \((-\partial V/\partial z)\) per unit length on the return wire and evaluating the \( z \)-component of (3.2) at the arbitrary point \( x = 0, y = c + r \) of the surface (as was done for \( V_\) in Sec. 2), we obtain

\[
U_\ = i \omega \Psi|_{x=0,y=c+r} - Z_i I.
\]

Finally, the impedance per unit length of the transmission line is deduced from (3.10) and (3.15) by

\[
Z = \frac{U_+ - U_-}{I}
\]

and clearly contains \( Z_i \) additively. In the sequel, we always use (3.15) with \( Z_i = 0 \), but one must remember that the independently specified value of \( Z_i \) should ultimately be added to (3.16).

Thus it results that both the linear current density and the impedance per unit length are deduced from the solution of the problem for \( \Psi(x, y) \) alone. The problem for the scalar potential \( V \) will therefore be disregarded in the rest of the paper, but the following remarks deal with its solution. Like \( V \), \( U = -\partial V/\partial z \) is harmonic. Since its boundary values \( U_+ \) and \( U_- \) on the strip and the return wire are constant, and determined by the solution of the problem for \( \Psi \), \( U \) itself is determined as the solution of a two-dimensional electrostatic problem. The potential \( V \) is then \(-zU(x, y) + W(x, y, z)\), where \( W \) is an arbitrary harmonic function such that \( \partial W/\partial z = 0 \) on both conductors. Now, the term \(-zU\) is proportional to \( I \) as are \( U_+ \) and \( U_- \) in (3.10) and (3.15) and determines the electric field generated by the currents \( \pm I \), whereas the term \( W \) corresponds to a separate electrostatic problem (one might have deposited arbitrary charge densities on the conductors) to be disregarded. With \( W = 0 \), the potential difference \( V_0 = -z(U_+ - U_-) \) induces a charge...
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\( q = CV_0 \) per unit length on the strip and a charge \(-q\) on the return wire, where \( C \) is the electrostatic capacitance per unit length determined in Sec. 2, and \( q \) is proportional to \( z \). The charges \( \pm q \) and the boundedness condition at large distance \((x^2 + y^2 \to \infty)\) determine the harmonic function \( U(x, y) \) everywhere, as proportional to the electrostatic potential of Sec. 2 plus an arbitrary constant. This constant is then deduced from the boundary values \( U_+ \) or \( U_- \), both clearly yielding the same result.

To conclude this section we discuss the physical meaning of the requirement for \( \Psi \) to vanish at large distance, imposed after (3.3), instead of a mere boundedness condition. If one replaces \( \Psi \) by \( \Psi + \Psi_\infty \), where \( \Psi_\infty \) is an arbitrary constant, the corresponding constant \( i\omega \Psi_\infty \) adds up in the boundary values \( U_\pm \) of (3.9) and (3.15), hence in \( U(x, y) \) everywhere. This does not affect the difference \( U_+ - U_- \) determining the impedance per unit length in (3.16), but one may question whether the values of \( U_+ \) and \( U_- \) can be interpreted as separate voltage drops per unit length on the go and return conductors. This is clearly correct under d.c. conditions, where \( i\omega \Psi_\infty \) vanishes in (3.9) and (3.15) and where the purely resistive voltage drops result from Ohm's law, separately for each conductor. This is also the case at high frequencies, or for good conductors, where the term \( j\sigma h \) in (3.7) is negligible with respect to \( i\omega \Psi \). Consequently, the boundary values (3.9) and (3.15) imply that \( \Psi \) takes constant values \( \Psi_+ \) and \( \Psi_- \) on either conductor. The high-frequency magnetic field is then purely external. With the condition \( \Psi_\infty = 0, \Psi_+ (-\Psi_-) \) is the flux per unit length encircling clockwise the go (return) conductor, whereas the total \( \Psi_+ - \Psi_- \) is the flux between the conductors, defining the (external) inductance per unit length; furthermore, the values \( i\omega \Psi_+ \) and \( i\omega \Psi_- \) define the inductive voltage drops per unit length on each conductor separately. It thus appears that the condition \( \Psi_\infty = 0 \) is the natural one to adopt in all cases, thus legitimizing the interpretations of \( U_\pm \) as individual voltage drops per unit length.

4. The high-frequency asymptotic lateral skin-effect

Using (1.1), we rewrite (3.10) as

\[
\frac{U_+}{i\omega} = \Psi|_{y=0} + \frac{\delta^2}{2ih} \left( \frac{\partial \Psi}{\partial y} \right)_{y=0-} - \left( \frac{\partial \Psi}{\partial y} \right)_{y=0+}. \tag{4.1}
\]

It thus appears that the constancy of (4.1) on the strip reduces to a much simpler condition (the constancy of \( \Psi|_{y=0} \) alone) when the coefficient \( \delta^2/h \) of the second term is vanishingly small. Since this coefficient has the dimension
of a length, and since \( h \) and \( \delta \) then disappear in the statement of the problem, the only remaining length parameters are \( a \) and \( c \). Consequently, the last term in (4.1) is negligible if both

\[
\delta^2 \ll ha; \quad \delta^2 \ll hc
\]  

(4.2)

hold, and these inequalities may be compatible with (1.2) over a wide frequency range. Since the problem then becomes frequency-independent, it characterizes the asymptotic behaviour of the lateral skin-effect at high frequencies.

With the boundary condition imposing the constancy of \( \Psi \) on the strip, the problem for \( \Psi(x, y) \) as specified in the context of (3.3) is equivalent to the electrostatic problem for \( V(x, y) \) solved in Sec. 2, and the solution is independent of the thickness profile \( h(x) \) and of the conductivity profile \( \sigma(x) \). The solution \( \Psi \) is simply obtained by replacing by \( \mu l/2\pi \) the coefficient \( q/2\pi \varepsilon \) of (2.4). In terms of the Joukowski coordinates, the solution is thus

\[
\Psi = \frac{\mu I}{2\pi} \log \left| \frac{\zeta - i e^\phi}{\zeta - i e^{-\phi}} \right|.
\]  

(4.3)

If one compares (3.4) with (2.15) and (2.19), it appears that \( j(x) \) is deduced from \( \sigma_+(x) + \sigma_-(x) \) by simply replacing \( q \) by \( I \). Thus it follows from (2.21) that the asymptotic linear current density is

\[
j(x) = \frac{Ic}{\pi} \frac{[(a^2 + c^2)/(a^2 - x^2)]^{1/2}}{e^2 + x^2}.
\]  

(4.4)

Since eq. (4.4) becomes infinite at the edges \( x = \pm a \) of the strip, a comparison of (3.4) with (4.1) shows that the factor multiplying \( \delta^2/2ih \) in (4.1) is also infinite, so that the approximation of (4.1) by \( \Psi \big|_{y=0} \) alone is \textit{a posteriori} illegitimate near the edges. This point is further discussed at the end of Sec. 9.

An expression equivalent to (4.4) but in terms of the variable \( x = a \cos \theta \) (\( 0 \leq \theta \leq \pi \)) can be obtained by adding to (2.18) the expression (2.19) with \( \theta \) changed into \( -\theta \). The sum involves the combination

\[
\frac{\sinh \phi}{\cosh \phi - \sin \theta} + \frac{\sinh \phi}{\cosh \phi + \sin \theta} = \frac{2 \sinh \phi \cosh \phi}{\cosh^2 \phi - \sin^2 \theta} = \frac{2 \sinh(2\phi)}{\cosh(2\phi) + \cos(2\theta)}
\]
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and the result is

\[
j(\theta) = \frac{I}{\pi a \sin \theta} \frac{\sinh(2\phi)}{\cosh(2\phi) + \cos(2\theta)}, \quad (0 \leq \theta \leq \pi).
\] (4.5)

The derivative with respect to \( \theta \) of the denominator of (4.5) vanishes at \( \theta = \pi/2 \), i.e. at \( x = 0 \), and at the roots of \( 2 - 3 \cos(2\theta) = \cosh(2\phi) \), equivalent to \( 3 \sin^2 \theta = \cosh^2 \phi \), hence for

\[
x = \pm \left[ \frac{1}{3} (2a^2 - c^2) \right]^{1/2}.
\] (4.6)

For \( c/a > \sqrt{2} \), the value (4.6) is imaginary and the function (4.4) has a single extremum, a minimum

\[
j = \frac{I}{\pi c} (1 + c^2/a^2)^{1/2}
\] (4.7)

at \( x = 0 \). For \( c/a < \sqrt{2} \), the function (4.4) has a maximum given by (4.7) for \( x = 0 \), and two minima

\[
j = \frac{3\sqrt{3} Ic}{2\pi(a^2 + c^2)}
\] (4.8)

occur at (4.6).

For a return at large distance (\( c = \infty \)), the result (4.4) becomes

\[
j(x) = \frac{I}{\pi(a^2 - x^2)^{1/2}},
\] (4.9)

yielding the same relative current distribution as the relative charge distribution (2.31). On the other hand, for a wide strip (\( a = \infty \)), (4.4) becomes

\[
j(x) = \frac{Ic}{\pi(c^2 + x^2)},
\] (4.10)

analogous to (2.29).

The impedance given by (3.16) becomes

\[
Z = \frac{i\omega}{I} (\Psi|_{y=0} - \Psi|_{x=0,y=c+r}).
\] (4.11)
The required values of $\Psi$ on the strip and on the return wire are simply obtained from (2.6) and (2.8), with the coefficient $q/2\pi c$ replaced by $\mu I/2\pi$. Thus we find

$$Z = i\omega L_\infty; \quad L_\infty = \frac{\mu}{2\pi} \log \left[ \frac{a}{r} \sinh(2\phi) \right], \quad (4.12)$$

where $L_\infty$ is the high-frequency inductance per unit length which is related to the electrostatic capacitance of (2.9) by the classical relation

$$L_\infty C = \varepsilon \mu. \quad (4.13)$$

The high-frequency resistance $R_\infty$ per unit length results from

$$R_\infty I^2 = \int_{-a}^{a} \frac{j^2(x)}{\sigma h} \, dx. \quad (4.14)$$

For finite $a$, it follows from (4.4) that $j^2(x)$ has non-integrable singularities at $x = \pm a$. Hence, the integral in (4.14) diverges if the conductivity and thickness profiles are such that $\sigma h$ stays finite at the edges of the strip, which is anyway required by (1.2). Consequently $R_\infty$ tends to infinity at high frequencies ($\omega \to \infty$) for all profiles, but its law of increase versus $\omega$ cannot be deduced from the elementary asymptotic theory of this section.

By contrast, for $a = \infty$ and constant $\sigma h$, eqs (4.10) and (4.14) yield

$$R_\infty = \frac{c^2}{\pi^2 \sigma h} \int_{-\infty}^{\infty} \frac{dx}{(c^2 + x^2)^2} = \frac{1}{2\pi \sigma hc}, \quad (4.15)$$

thus showing that the proximity effect is equivalent to the constriction of the infinite width of the strip into a finite width $2\pi c$.

For small $\phi$, or equivalently for large $a$, the expression (4.12) reduces by (2.3) to

$$L_\infty = \frac{\mu}{2\pi} \log \left( \frac{2c}{r} \right), \quad (4.16)$$

related to (2.14) by (4.13), and we have

$$R_\infty = \frac{1}{\mu \sigma h} \frac{\partial L_\infty}{\partial c}, \quad (4.17)$$

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a result reminiscent of Wheeler's incremental rule\(^6\). We now establish (4.17) by a perturbation analysis avoiding the integration involved in (4.14).

By analogy with (2.13), the solution (4.3) for \(a = \infty\) becomes

\[
\Psi = \frac{\mu I}{4\pi} \log \left[ \frac{x^2 + (y - c)^2}{x^2 + (y + c)^2} \right], \quad (y \geq 0)
\]

while \(\Psi = 0\) for \(y \leq 0\); hence, \(U_+ = 0\) by (4.1). Let us now consider the boundary value problem for \(\Psi\) in the case where \(\delta^2/2h\) is small but non-zero. Then, within the first order in \(\delta^2/2h\), the solution (4.18) for \(\delta = 0\) must be replaced by a solution of the form

\[
\Psi' = \Psi + \frac{\delta^2}{2ih} \chi, \quad (y \geq 0)
\]

where \(\chi\) is a regular harmonic function to be determined such that the constancy of (4.1) with \(\Psi\) replaced by \(\Psi'\) be satisfied within the first order. By use of the derivatives

\[
\frac{\partial \Psi}{\partial y} \bigg|_{y=0^+} = \frac{\mu I}{4\pi} \left( \frac{-4c}{x^2 + c^2} \right),
\]

\[
\frac{\partial \Psi}{\partial y} \bigg|_{y=0^-} = 0,
\]

the solution for \(\chi\) is found to be

\[
\chi = \frac{\mu I}{2\pi} \frac{\partial}{\partial c} \log \left[ \frac{1}{x^2 + (y + c)^2} \right] = \frac{4(\mu I)}{2\pi} \left[ \frac{-4(y + c)}{x^2 + (y + c)^2} \right].
\]

Clearly, the function (4.22) is harmonic and takes the same value as the function (4.20) for \(y = 0\). Thus the boundary condition (4.1) is satisfied for \(\Psi'\) within the first order, and we still have \(U_+ = 0\). The impedance deduced from (3.15) and (3.16) then reduces to

\[
Z = -\frac{U_-}{I} = -\frac{i\omega}{I} \Psi' \bigg|_{x=0, y=c+r}.
\]
If one replaces $\Psi'$ by (4.19), the term $\Psi$ yields the reactance $\omega L_\infty$, where $L_\infty$ is given by (4.16), whereas the term in $\chi$ resulting from (4.22), evaluated at $x = 0$, $y = c$, yields eq. (4.17).

Finally, we recall that the high-frequency asymptotic behaviour treated in the present section is restricted to the lateral effect where condition (1.2) holds. At higher frequencies, when condition (1.2) is violated, the current density is no longer independent of $y$ and depth penetration begins. At frequencies such that one has $\delta \ll h$, the depth penetration reaches its asymptotic stage and the current density separates into two densities $j_+$ and $j_-$ at the two sides of the strip. At that stage eq. (3.4) no longer holds and the densities $j_\pm$ are determined by $j_\pm = \mp (1/\mu)(\partial \Psi/\partial y)_{y=0}^\pm$ as in (2.15) and (2.19), because one has $\Psi = 0$ inside the strip. At that second high-frequency asymptotic stage the electromagnetic problem becomes strictly equivalent to the electrostatic problem and the expressions for $j_\pm$ are analogous to those in (2.21) for $\sigma_\pm$. Needless to say, this second asymptotic stage is still compatible with the quasi-stationary approximation over a wide frequency range.

5. The lateral skin-effect in an infinitely wide strip

In this section we pursue the solution of the problem for $\Psi$ stated in Sec. 3, but restrict ourselves to the simplest case of an infinitely wide strip ($a = \infty$) of uniform thickness $h$ and conductivity $\sigma$.

It follows from the discussion around (3.3) that the solution for $\Psi$ is of the form

$$\Psi = \frac{\mu I}{4\pi} \left\{ \log \left[ \frac{x^2 + (y - c)^2}{x^2 + (y + c)^2} \right] + \int_0^\infty F(\alpha) e^{-\alpha y} \cos(\alpha x) \, d\alpha \right\}, \quad (y \geq 0)$$

$$\Psi = \frac{\mu I}{4\pi} \int_0^\infty F(\alpha) e^{\alpha y} \cos(\alpha x) \, d\alpha, \quad (y \leq 0).$$

This is justified as follows: the numerator under the logarithm of (5.1) yields the potential of the thin wire with the singular part (3.3) and the denominator is chosen so as to make $\Psi$ vanish at large distance; among the various possibilities ensuring this, the choice of a singularity at the image point as in (2.13) leads to the simplest form of the boundary condition to be further established; the integral in (5.1) is the general harmonic function, even in $x$, and vanishing at large distance ($y = +\infty$); similarly, the integral (5.2) is the general harmonic function vanishing for $y = -\infty$, and the
appearance of the same arbitrary function $F(\alpha)$ in (5.1) and (5.2) results from the continuity requirement on $\Psi$ across $y = 0$.

By inserting (5.1) and (5.2), the boundary condition (3.10) becomes

$$U_+ = \frac{I}{4\pi h} \left[ \frac{4c}{x^2 + c^2} + \int_0^\infty (2\alpha + i\omega\sigma h)F(\alpha)\cos(\alpha x)\,d\alpha \right]$$

and $U_+$ must be independent of $x$. From the elementary integral

$$\frac{c}{x^2 + c^2} = \int_0^\infty e^{-\alpha x} \cos(\alpha x)\,d\alpha$$

it is seen that one obtains the constant value

$$U_+ = 0$$

for

$$F(\alpha) = \frac{4e^{-\alpha c}}{2\alpha + i\omega\sigma h}.$$ (5.5)

Thus the solutions (5.1) and (5.2) are now uniquely determined.

The voltage drop per unit length $U_-$ on the surface of the return wire is simply the value of $i\omega\Psi$ (cf. (3.15)) and is obtained by setting $x = 0$, $y = c + r$ in the expression under the logarithm of (5.1), but $x = 0$, $y = c$ elsewhere. One thus obtains

$$U_- = \frac{i\omega\mu I}{4\pi} \left[ \log\left( \frac{r^2}{4c^2} \right) + \int_0^\infty F(\alpha)e^{-\alpha c}\,d\alpha \right].$$ (5.6)

By using (5.4)–(5.6) in (3.16), the impedance per unit length is

$$Z = \frac{i\omega\mu}{2\pi} \left[ \log\left( \frac{2c}{r} \right) + 2 \int_0^\infty \frac{e^{-2\alpha c}}{2\alpha + i\omega\sigma h}\,d\alpha \right].$$ (5.7)

The first term of (5.7) coincides with the high-frequency inductance $L_\infty$ in (4.16). Any skin-effect impedance $Z = R + i\omega L$ has the nature of the impedance of a (distributed) RL-circuit, so that the inductance $L$ is a monotonically decreasing function of frequency having a minimum value $L_\infty$. 

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at infinity. Since there is no internal magnetic field at high frequencies, $L_\infty$ is then the external inductance. It is not true, however, that $L_\infty$ is the external inductance at intermediate frequencies (ref. 2, Sec. 1 and Appendix A), so that the difference

$$Z_m = Z - i\omega L_\infty$$  \hspace{1cm} (5.8)

is not the internal impedance, but just the minimum-reactance impedance.

By (5.7), (5.8) becomes

$$Z_m = \frac{i\omega\mu}{\pi} \int_0^\infty \frac{e^{-2a\alpha}}{2\alpha + i\omega\mu\sigma h} \, d\alpha$$  \hspace{1cm} (5.9)

from which it appears that, at high frequency ($\omega \rightarrow \infty$), $Z_m$ tends to the value $R_\infty$ given by (4.15). Introducing the normalized complex frequency

$$p = i\omega\mu\sigma h c,$$  \hspace{1cm} (5.10)

changing $\alpha$ into $\alpha/2c$ in (5.9) and using (4.15), one obtains

$$\frac{Z_m}{R_\infty} = p \int_0^\infty \frac{e^{-a}}{p + \alpha} \, d\alpha.$$  \hspace{1cm} (5.11)

In terms of the exponential integral

$$E_1(p) = \int_p^\infty \frac{e^{-\alpha}}{\alpha} \, d\alpha$$  \hspace{1cm} (5.12)

in the notation of Abramowitz and Stegun 7), we have

$$p \int_0^\infty \frac{e^{-a}}{p + \alpha} \, d\alpha = p \exp \int_p^\infty \frac{e^{-a}}{\alpha} \, d\alpha = p \exp E_1(p)$$  \hspace{1cm} (5.13)

so that expression (5.11) becomes

$$\frac{Z_m}{R_\infty} = p \exp E_1(p).$$  \hspace{1cm} (5.14)

The continued fraction expansion (ref. 7, formula 5.1.22) of the normalized
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![Ladder network representation of the impedance](image)

Fig. 2. Ladder network representation of the impedance \( Z_m/R_\infty \) from (5.14).

The admittance corresponding to (5.14) is

\[
\frac{e^{-p}}{p E_1(p)} = 1 + \frac{1}{p+1} + \frac{1}{p+2} + \frac{1}{p+3} + \cdots
\]

(5.15)

which shows that \( Z_m/R_\infty \) admits the ladder network representation of fig. 2. It is also recognized that, at high frequencies, the resistance tends to \( R_\infty \) and the inductance to 0. Since the expression (5.11) is a Stieltjes integral defining a positive real function of the RL-type, the resistance increases monotonically with frequency from 0 under d.c. conditions to \( R_\infty \) at infinity, whereas the inductance decreases monotonically from an infinite value under d.c. conditions to 0 at high frequency.

From the series expansion

\[
E_1(p) = -\gamma - \log p - \sum_{n=1}^{\infty} \frac{(-1)^n p^n}{n \, n!}
\]

(5.16)

where \( \gamma = 0.57721 \ldots \) is Euler's constant, one deduces the principal value of (5.14) for small \( |p| \) as

\[
\frac{Z_m}{R_\infty} = -p(\gamma + \log p).
\]

(5.17)

This yields \( Z_m = R_0 + i\omega L_0 \) with

\[
R_0 = \omega \mu / 4,
\]

(5.18)

\[
L_0 = -\frac{\mu}{2\pi} [\gamma + \log(\omega \mu \sigma c)]
\]

(5.19)

by (5.10) and (4.15), so that the resistance vanishes under d.c. conditions (because of the infinite width of the strip), whereas the inductance is infinite.
The linear current density on the strip is computed using (3.14). Using (5.2), (5.4) and (5.5), we obtain

\[
j(x) = \frac{pI}{\pi} \int_{0}^{\infty} \frac{e^{-ax} \cos(\alpha x)}{p + 2\alpha c} \, d\alpha.
\]  

(5.20)

It is easily checked by using Fourier's theorem that condition (3.5) is satisfied. By (5.13), with \( a \) changed into \( sa \) and \( p \) changed into \( sp/2c \), we have

\[
\int_{0}^{\infty} \frac{e^{-sx}}{p + 2\alpha c} \, d\alpha = \frac{1}{2c} \exp \left( \frac{sp}{2c} \right) E_{1} \left( \frac{sp}{2c} \right); \quad \text{Re} \, s > 0.
\]  

(5.21)

The integral in eq. (5.20) is evaluated by means of eq. (5.21) with \( s = c \pm ix \); one thus obtains

\[
j(x) = \frac{pI}{4\pi c} \left\{ \exp \left[ \frac{p(c + ix)}{2c} \right] E_{1} \left( \frac{p(c + ix)}{2c} \right) + \exp \left[ \frac{p(c - ix)}{2c} \right] E_{1} \left( \frac{p(c - ix)}{2c} \right) \right\}.
\]  

(5.22)

Both results (5.20) and (5.22) show that the asymptotic expression of \( j(x) \) for large \( |p| \) is given by (4.10).

6. The general electromagnetic problem for a strip of finite width

We now pursue the solution of the problem for \( \Psi \) stated in Sec. 3 for the general case: the strip has a finite width \( 2a \) and arbitrary thickness profile \( h(x) \) and conductivity profile \( \sigma(x) \). As in Sec. 2, we use a hybrid system of coordinates: both the complex variable \( \zeta \) of the Joukowski transformation and the polar coordinates \( \rho, \theta \) of (2.1). From the text following (2.1) and (2.17), we recall that the upper (lower) side of the strip is described by \( x = a \cos \theta, \, y = 0+ \) (\( y = 0- \)) where \( 0 \leq \theta \leq \pi \) \( (-\pi \leq \theta \leq 0) \).

The general solution for \( \Psi \) is of the form

\[
\Psi = \frac{\mu I}{2\pi} \log \left| \frac{\zeta - i e^{\phi}}{\zeta - i e^{-\phi}} \right| + \sum_{n=1}^{\infty} A_{n} \rho^{-2n} \cos(2n\theta),
\]  

(6.1)

which will now be justified. The logarithmic term (4.3) of (6.1) supplies the singularity required by (3.3) and vanishes at large distance \( (x^{2} + y^{2} \to \infty) \); the series in (6.1) involving the undetermined coefficients \( A_{n} \) represents the most general function, harmonic in \( |\zeta| \) \( > 1 \) and vanishing for \( \rho \to \infty \), of period \( 2\pi \) in \( \theta \) and having even symmetry in \( x \) and \( y \) in accordance
with the symmetry of the problem: the even symmetry in $y$ implies that the series in (6.1) must be even in $\theta$ and thus contains no terms in $\sin(n\theta)$, the even symmetry in $x$ implies that the series must be invariant under a replacement of $\theta$ by $\pi - \theta$ and this excludes cosines of odd multiples of $\theta$.

On the strip, the logarithmic term of (6.1) takes the constant value $\mu I \phi/2\pi$, by analogy with (2.6), and the solution (6.1) takes the value

$$\Psi|_{y=0} = \frac{\mu I}{2\pi} \left[ \phi + \sum_{n=1}^{\infty} A_n \cos(2n\theta) \right].$$

(6.2)

We now establish that one has

$$-\frac{\partial \Psi}{\partial y} |_{y=0^+} = \frac{\mu I}{2\pi a \sin \theta} \left[ \frac{\sinh \phi}{\cosh \phi - \sin \theta} + 2 \sum_{n=1}^{\infty} nA_n \cos(2n\theta) \right], \quad (0 \leq \theta \leq \pi).$$

(6.3)

The first term of the right-hand side of (6.3) is the analogue of (2.18), whereas the second term is readily evaluated as in eq. (2.17) by considering that $\rho^{-2n} \cos(2n\theta)$ is the real part of $\zeta^{-2n}$, hence by

$$\frac{\partial}{\partial y} \text{Re}(\zeta^{-2n}) = -\text{Im} \left( \frac{d\zeta^{-2n}}{d\zeta} \right) = \frac{4n}{a} \text{Im} \left( \frac{\zeta^{-2n-1}}{1 - \zeta^{-2}} \right).$$

For $\rho = 1$, this becomes

$$\frac{4n}{a} \text{Im} \left( e^{-2i\theta} \right) = -\frac{2n \cos(2n\theta)}{a \sin \theta}$$

and this justifies (6.3). Similarly we have

$$-\frac{\partial \Psi}{\partial y} |_{y=0^-} = \frac{\mu I}{2\pi a \sin \theta} \left[ -\frac{\sinh \phi}{\cosh \phi + \sin \theta} - 2 \sum_{n=1}^{\infty} nA_n \cos(2n\theta) \right], \quad (-\pi \leq \theta \leq 0),$$

(6.4)

where the first term is the analogue of (2.19), whereas the sign of the second term is changed in (6.4) with respect to (6.3) because of the change of signs in the left-hand sides. In order to compute (6.4) as a function of $\theta$ in $0 \leq \theta \leq \pi$ we must change $\theta$ into $-\theta$; this yields

$$-\frac{\partial \Psi}{\partial y} |_{y=0^-} = \frac{\mu I}{2\pi a \sin \theta} \left[ \frac{\sinh \phi}{\cosh \phi + \sin \theta} + 2 \sum_{n=1}^{\infty} nA_n \cos(2n\theta) \right], \quad (0 \leq \theta \leq \pi).$$

(6.5)
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In the summation of (6.3) and (6.5) the first terms on the right-hand sides combine as in the equation preceding (4.5). Their sum is next represented by the elementary Fourier series

\[
\frac{\sinh(2\phi)}{\cosh(2\phi) + \cos(2\theta)} = 1 + 2 \sum_{n=1}^{\infty} \tau^n \cos(2n\theta), \tag{6.6}
\]

where we have set

\[-e^{-2\phi} = \tau. \tag{6.7}\]

In this manner we obtain

\[
\left. \frac{\partial \Psi}{\partial y} \right|_{y=0^-} - \left. \frac{\partial \Psi}{\partial y} \right|_{y=0^+} = \frac{\mu I}{\pi a \sin \theta} \left[ 1 + 2 \sum_{n=1}^{\infty} (\tau^n + nA_n) \cos(2n\theta) \right], \tag{6.8}
\]

where \(0 \leq \theta \leq \pi\).

Finally, by inserting (6.2) and (6.8), the boundary condition (3.10) becomes

\[
\frac{i\omega \mu I \phi}{2\pi} - U_+ + \frac{i\omega I}{2\pi} \sum_{n=1}^{\infty} A_n \cos(2n\theta)
\]

\[
+ \frac{I}{\pi a h_0 \sin \theta} \left[ 1 + 2 \sum_{n=1}^{\infty} (\tau^n + nA_n) \cos(2n\theta) \right] = 0, \quad (0 \leq \theta \leq \pi). \tag{6.9}
\]

Multiplying (6.9) by \(2\pi/i\omega \mu I\) and defining the constant

\[
A_0 = 2 \left( \phi - \frac{2\pi U_+}{i\omega \mu I} \right), \tag{6.10}
\]

we rewrite (6.9) as

\[
\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\theta) + \frac{2}{i\omega \mu h_0 \sin \theta} \left[ 1 + 2 \sum_{n=1}^{\infty} (\tau^n + nA_n) \cos(2n\theta) \right] = 0, \tag{6.11}
\]

and the coefficients \(A_n\) must be determined by requiring that eq. (6.11) holds for all \(\theta\) in the range \(0 \leq \theta \leq \pi\). This determination depends on the assumed profiles \(h(\theta)\) and \(\sigma(\theta)\) and is postponed until Sec. 7. We now continue the general discussion so as to obtain the impedance per unit length.
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and the linear current density on the strip in terms of the temporarily undetermined coefficients $A_n$.

Using (6.10) we have

$$U_+ = \frac{i\omega \mu l}{2\pi} \left( \phi - \frac{1}{2} A_0 \right). \quad (6.12)$$

On the other hand, referring to (3.15), $U_-$ is the value of $i\omega y'$ on the surface of the thin wire. Here the logarithmic term of (6.1) yields

$$\frac{i\omega \mu l}{2\pi} \left\{ \log \left[ \frac{r}{a \sinh(2\phi)} \right] + \phi \right\}$$

by analogy with (2.8), whereas the remaining terms of (6.1) can be evaluated at $x = 0$, $y = c$, i.e. at $\rho = e^\phi$, $\theta = \pi/2$. One thus obtains, using (6.7),

$$U_- = \frac{i\omega \mu l}{2\pi} \left\{ \log \left[ \frac{r}{a \sinh(2\phi)} \right] + \phi + \sum_{n=1}^{\infty} A_n \tau^n \right\}. \quad (6.13)$$

Finally, the impedance per unit length (3.16) becomes $Z = i\omega L_\infty + Z_m$, where $L_\infty$ is given by (4.12) and where

$$Z_m = -\frac{i\omega \mu}{2\pi} \left( \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \tau^n \right). \quad (6.14)$$

From (6.2) and (6.12) the linear current density given by (3.14) is found to be

$$j(\theta) = -\frac{i\omega \mu \sigma h l}{2\pi} \left[ \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\theta) \right]. \quad (6.15)$$

The alternative expression

$$j(\theta) = \frac{l}{\pi a \sin \theta} \left[ 1 + 2 \sum_{n=1}^{\infty} (\tau^n + nA_n) \cos(2n\theta) \right] \quad (6.16)$$

deduced from (3.4) and (6.8) is also of interest. The equivalence of (6.15) and (6.16) results from (6.11). Since $x = a \cos \theta$, the identity (3.5)
becomes

$$a \int_{0}^{\pi} j(\theta) \sin \theta \, d\theta = I$$

(6.17)

and is clearly satisfied by (6.16).

7. The lateral skin-effect in a thin strip of elliptic cross-section

In order to determine the coefficients $A_n$ in (6.11) one must multiply that equation by the factor $\sigma h \sin \theta$ depending on $\theta$, expand this factor in a Fourier series holding for $0 \leq \theta \leq \pi$ and set equal to zero the coefficients of $\cos(2n\theta)$ in the resulting expression. The Fourier series for $\sigma h \sin \theta$ depends on the assumed profiles $\sigma(\theta)$ and $h(\theta)$. If $\sigma$ and $h$ are constant this Fourier series is

$$\sin \theta = \frac{2}{\pi} \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{4n^2 - 1} \right], \quad (0 \leq \theta \leq \pi),$$

(7.1)

and the resulting equations for the coefficients $A_n$ form an infinite linear system which cannot be solved explicitly.

In contrast, if one assumes that the cross-section of the strip is a thin ellipse of semi-axis $a$ in the $x$-direction and $b$ in the $y$-direction, with $b \ll a$, one has

$$h(\theta) = 2b \sin \theta, \quad (0 \leq \theta \leq \pi),$$

(7.2)

so that

$$\sigma h \sin \theta = 2\sigma b \sin^2 \theta = \sigma b[1 - \cos(2\theta)].$$

(7.3)

For constant $\sigma$, (7.3) involves only two terms, compared with the infinity of terms in eq. (7.1). Consequently the infinite linear system in the unknowns $A_n$ resulting from (6.11) has a tridiagonal matrix in the case of (7.3), instead of a full matrix in the case of (7.1), and can be inverted explicitly. In the rest of the paper, we thus deal exclusively with the elliptic strip. Since $x = a \cos \theta$, the thickness profile (7.2) is

$$h(x) = 2b(1 - x^2/a^2)^{1/2}$$

(7.4)
and tends to the constant thickness

\[ h = 2b \]  \hspace{1cm} (7.5)

for a strip of infinite width \((a = \infty)\). It is verified in the Appendix that the solutions further obtained for the elliptic strip tend to those obtained in Sec. 5 for \(a = \infty\) and constant \(\sigma h\).

Introducing the normalized frequency

\[ k = \frac{1}{2} \omega \mu \sigma ab \]  \hspace{1cm} (7.6)

and using (7.3), one rewrites (6.11) as

\[
\left[ 1 - \cos(2\theta) \right] \left[ \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\theta) \right] + \frac{1}{k} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} (\tau^n + nA_n) \cos(2n\theta) \right] = 0. \hspace{1cm} (7.7)
\]

Since

\[
\left[ 1 - \cos(2\theta) \right] \left[ \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\theta) \right] = \frac{1}{2} A_0 - \frac{1}{2} A_1 - \sum_{n=1}^{\infty} (\frac{1}{2} A_{n-1} - A_n + \frac{1}{2} A_{n+1}) \cos(2n\theta) \hspace{1cm} (7.8)
\]

the linear equations in the coefficients \(A_n\) resulting from (7.7) are

\[
A_0 - A_1 + \frac{1}{k} = 0, \hspace{1cm} (7.9)
\]

\[
A_{n-1} - 2A_n + A_{n+1} - \frac{2}{k} (\tau^n + nA_n) = 0, \hspace{1cm} (n = 1, 2, \ldots). \hspace{1cm} (7.10)
\]

Setting

\[
A_n = -\frac{2}{k} F_n, \hspace{1cm} (n = 0, 1, 2, \ldots), \hspace{1cm} (7.11)
\]

one changes the system of equations (7.9) and (7.10) into

\[
F_0 - F_1 = \frac{1}{2}, \hspace{1cm} (7.12)
\]
The resulting recurrence equations, written in matrix form, are

\[
\begin{bmatrix}
1 & -1 & 0 \\
-1 & 2\left(1+\frac{1}{k}\right) & -1 \\
0 & \cdots & 2\left(1+\frac{2}{k}\right) & -1 \\
\end{bmatrix}
\begin{bmatrix}
F_0 \\
F_1 \\
F_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} \\
\tau \\
\tau^2 \\
\vdots
\end{bmatrix},
\]  

(7.14)

where the large 0s stand for zero entries. The solution of the infinite linear system (7.14) can be represented by

\[
F_n = \sum_{s=0}^{\infty} \varepsilon_s M_{ns} \tau^s, 
\]  

(7.15)

where \(\varepsilon_s\) is one-half of the Neumann factor, i.e. \(\varepsilon_s = 1\) for \(s \neq 0\), \(\varepsilon_0 = \frac{1}{2}\), and where the coefficients \(M_{ns}\) are the entries of the inverse of the matrix occurring in (7.14). The \(s\)th column vector \(M_{ns}\) \((n = 0, 1, 2, \ldots)\) is determined as the solution of the system

\[
M_{0s} - M_{1s} = \delta_{0s},
\]  

(7.16)

\[
-M_{n-1,s} + 2\left(1 + \frac{n}{k}\right) M_{ns} - M_{n+1,s} = \delta_{ns}, 
\]  

(7.17)

where \(\delta_{ns} = 1\) if \(n = s\) and \(\delta_{ns} = 0\) if \(n \neq s\) (Kronecker’s symbol). For \(n \neq s\), eq. (7.17) reduces to

\[
-M_{n-1,s} + 2\left(1 + \frac{n}{k}\right) M_{ns} - M_{n+1,s} = 0,
\]  

(7.18)

which is recognized as the recurrence relation for the Bessel functions \(J_{n+k}(k)\) and \(Y_{n+k}(k)\). In the following we abbreviate these as \(J_n\) and \(Y_n\).

Consider first the case \(s > 0\). Then the solution of (7.17) for \(n \neq s\), i.e. of (7.18), is

\[
M_{ns} = \alpha J_n + \beta Y_n, 
\]  

(0 \leq n \leq s)

(7.19)
The solution $Y_n$ being excluded in (7.20) because it tends to infinity for large $n$. Substitution of (7.19) with $n = 0$ and $n = 1$ into (7.16) with $\delta_{0s} = 0$ yields

$$\alpha(J_0 - J_1) + \beta(Y_0 - Y_1) = 0. \quad (7.21)$$

Also, the two results (7.19) and (7.20) for $n = s$ should be identical, hence

$$(\alpha - \gamma)J_s + \beta Y_s = 0. \quad (7.22)$$

Finally, eq. (7.17) written for $n = s$, and using (7.19) for $n = s - 1$ and (7.20) for $n = s$ and $n = s + 1$, becomes

$$-\alpha J_{s-1} - \beta Y_{s-1} + 2\left(1 + \frac{s}{k}\right)\gamma J_s - \gamma J_{s+1} = 1. \quad (7.23)$$

By using the recurrence relation for $J_s$ this simplifies into

$$(\gamma - \alpha)J_{s-1} - \beta Y_{s-1} = 1. \quad (7.23)$$

By using the Wronskian identity

$$J_s Y_{s-1} - J_{s-1} Y_s = \frac{2}{\pi k} \quad (7.24)$$

one obtains the solution of (7.22) and (7.23) as

$$\beta = -\frac{\pi k}{2} J_s; \quad \gamma - \alpha = -\frac{\pi k}{2} Y_s. \quad (7.25)$$

Now, eq. (7.21) yields

$$\alpha = -\beta \frac{Y_0 - Y_1}{J_0 - J_1} = \frac{\pi k}{2} \frac{Y_0 - Y_1}{J_0 - J_1}$$

and the second of eqs (7.25) then gives

$$\gamma = -\frac{\pi k}{2} \frac{Y_s(J_0 - J_1) - J_s(Y_0 - Y_1)}{J_0 - J_1}. \quad (7.26)$$
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We next consider the case $s = 0$. The solution of (7.18) is

$$M_{n0} = \gamma J_n, \quad (n = 0, 1, 2, \ldots)$$

as in (7.20). On the other hand, eq. (7.16) for $s = 0$ yields

$$\gamma = \frac{1}{J_0 - J_1}. \quad (7.28)$$

Owing to (7.24) for $s = 1$, the result (7.28) coincides with the particular case of (7.26) for $s = 0$.

Finally, inserting into (7.19) and (7.20) the coefficients resulting from (7.25) and (7.26) we obtain

$$M_{ns} = \frac{\pi k}{2} \frac{J_n}{J_0 - J_1} [(Y_0 - Y_1)J_n - (J_0 - J_1)Y_n], \quad (0 \leq n \leq s), \quad (7.29)$$

$$M_{ns} = \frac{\pi k}{2} \frac{J_n}{J_0 - J_1} [(Y_0 - Y_1)J_n - (J_0 - J_1)Y_n], \quad (n \geq s \geq 0), \quad (7.30)$$

and these results hold down to $s = 0$. One thus has $M_{ns} = M_{sn}$, as expected for the inverse of the symmetric matrix of (7.14).

We now establish a generalization of the Wronskian identity showing that the expressions between square brackets in (7.29) and (7.30) are polynomials in $k^{-1}$ related to Lommel's polynomials. Consider the $(s \times s)$ determinant

$$D_s = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2\left(1 + \frac{1}{k}\right) & -1 \\ \vdots & \vdots & \ddots \\ -1 & 2\left(1 + \frac{2}{k}\right) & -1 \\ 0 & \cdots & -1 & 2\left(1 + \frac{s-1}{k}\right) \end{vmatrix} \quad (7.31)$$
which is the principal minor of order $s$ of the matrix of (7.14). By the Laplace expansion based on the last row of (7.31) one establishes the recurrence relation

$$D_{s+1} = 2 \left(1 + \frac{s}{k}\right)D_s - D_{s-1}, \quad (s = 2, 3, \ldots). \quad (7.32)$$

Since $D_1 = 1, D_2 = 2(1 + 1/k) - 1$, it appears that eq. (7.32) holds down to $s = 1$ if one sets $D_0 = 1$. Now eq. (7.32) is identical to the recurrence relation for the Bessel functions $J_s$ and $Y_s$. Expressing $D_s$ as a linear combination of these functions, one determines the coefficients of the combination by the initial conditions $D_0 = D_1 = 1$. By using the Wronskian identity (7.24) for $s = 1$, one finally obtains

$$D_s = \frac{\pi k}{2} [(Y_0 - Y_1)J_s - (J_0 - J_1)Y_s]. \quad (7.33)$$

One thus simplifies eqs (7.29) and (7.30) into

$$M_{ns} = \frac{J_n D_n}{J_0 - J_1}, \quad (0 \leq n \leq s); \quad (7.34a)$$

$$M_{ns} = \frac{J_n D_s}{J_0 - J_1}, \quad (n \geq s \geq 0). \quad (7.34b)$$

Also, the denominator in (7.34) is

$$J_0 - J_1 = J_k(k) - J_{k+1}(k) = J'_k(k). \quad (7.35)$$

Substituting (7.11) into (6.14), dividing by the d.c. resistance per unit length

$$R_0 = \frac{1}{\pi \sigma ab} \quad (7.36)$$

of the elliptic strip and using (7.6), we obtain

$$\frac{Z_m}{R_0} = 2F_0 + 4 \sum_{n=1}^{\infty} F_n \tau^n. \quad (7.37)$$
Defining the column-vector $t$ by its transpose row-vector

$$t' = [\frac{1}{2}, \tau, \tau^2, \ldots]$$

we have, by use of (7.15),

$$\frac{1}{2}F_0 + \sum_{n=1}^{\infty} F_n \tau^n = t'Mt$$

where $M$ is the matrix of entries given by (7.34). Thus expression (7.37) becomes

$$\frac{Z_m}{R_0} = 4t'Mt.$$  

Consider the ladder network of fig. 3. Call $F_n$ the potential of node $n$ with respect to the ground and inject an external current $\epsilon_n \tau^n$ into node $n$. Equations (7.14) are then the Kirchhoff current relations at the nodes, so that the matrix of (7.14) is the admittance matrix of the infinite $n$-port and its inverse $M$ is the impedance matrix. Since the network is passive, $M$ is a positive real matrix, so that the impedance (7.40) is a positive real function. The substitution of (7.34) into (7.40) yields an explicit expression for $Z_m$ as an infinite series in powers of $\tau$ whose coefficients are finite sums involving rational and Bessel functions of argument $k$; however, it does not seem possible to simplify these coefficients further.
Using (7.2), (7.6) and (7.11), the linear current density (6.15) becomes

\[ j(\theta) = \frac{4I}{\pi a} \sin \theta \left[ F_0 + 2 \sum_{n=1}^{\infty} F_n \cos(2n\theta) \right] \]  (7.41)

which again becomes a double sum by use of (7.15) and (7.34) and no further simplification seems possible. In any case, eq. (7.41) shows that \( j(\theta) \) vanishes at the edges \( (\theta = 0 \text{ and } \theta = \pi) \) of the strip. The alternative expression resulting from (6.16) is

\[ j(\theta) = \frac{I}{\pi a} \sin \theta \left[ \frac{\sinh(2\phi)}{\cosh(2\phi) + \cos(2\theta)} - 4 \sum_{n=1}^{\infty} nF_n \cos(2n\theta) \right] \]  (7.42)

owing to (6.6).

For a return at large distance \( (c = \infty, \phi = \infty) \), we have \( \tau = 0 \) from (6.7), and expression (7.40) reduces to

\[ \frac{Z_m}{R_0} = M_{00} = \frac{J_k(k)}{J_k(k)} \]  (7.43)

using (7.34) and (7.35), and this agrees with a previous result (ref. 5, eq. (17)). Also the result (7.15) then reduces to

\[ F_n = \frac{1}{2} M_{00}, \]

and hence, using (7.34) and (7.35), to

\[ F_n = \frac{1}{2} \frac{J_n}{J_0 - J_1} = \frac{1}{2} \frac{J_{n+k}(k)}{J_k(k)} \]

Finally, eqs (7.41) and (7.42) reduce to the known results (ref. 5, eqs (19) and (20))

\[ j(\theta) = \frac{2I \sin \theta}{\pi a J_k^\prime(k)} \left[ J_k(k) + 2 \sum_{n=1}^{\infty} J_{n+k}(k) \cos(2n\theta) \right], \]  (7.45)

\[ j(\theta) = \frac{I}{\pi a \sin \theta} \left[ 1 - \frac{2}{k J_k^\prime(k)} \sum_{n=1}^{\infty} n J_{n+k}(k) \cos(2n\theta) \right]. \]  (7.46)
8. Low-frequency approximations

Under d.c. conditions, the true current density (A m$^{-2}$) in the elliptic strip is uniform and is thus equal to $I$ divided by the ellipse area $\pi ab$. Because of the non-uniform thickness $h$ given by (7.2), the linear current density is not uniform and is given by

$$j(\theta) = \frac{hI}{\pi ab} = \frac{2I}{\pi a} \sin \theta, \quad (0 \leq \theta \leq \pi),$$

(8.1)

and we first check that eq. (7.41) yields that value for $k=0$.

Consider the system of equations (7.12) and (7.13) in the case that $|k|$ is small but non-zero. From (7.13) multiplied by $k$, we deduce the solution, to the first order in $k$,

$$F_n = \frac{k}{2n} \tau^n, \quad (n = 2, 3, \ldots)$$

(8.2)

which is not valid for $n = 0$ and $n = 1$ because of (7.12). Eliminating $F_0$ from (7.12) and (7.13) for $n = 1$, we obtain

$$\left(1 + \frac{2}{k}\right)F_1 - F_2 = \tau + \frac{1}{2}.$$  

Hence, with the value of $F_2$ given by (8.2), we have to the first order in $k$,

$$F_1 = \frac{k}{2} \left(\tau + \frac{1}{2}\right)$$

(8.3)

and next by (7.12)

$$F_0 = \frac{1}{2} + \frac{k}{2} \left(\tau + \frac{1}{2}\right).$$

(8.4)

For $k = 0$, eqs (8.2) to (8.4) yield $F_0 = \frac{1}{2}, F_n = 0$ for $n \neq 0$, so that eq. (7.41) indeed reduces to (8.1).

In order to obtain the value of the inductance under d.c. conditions, one
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must evaluate (7.37) to the first order in \( k \). One obtains

\[
\frac{Z_m}{R_0} = 1 + k \left( \tau + \frac{1}{2} \right) + 2k \tau \left( \tau + \frac{1}{2} \right) + 2k \sum_{n=2}^{\infty} \frac{\tau^{2n}}{n}
\]

\[
= 1 + \frac{k}{2} + 2k \left( \tau + \sum_{n=1}^{\infty} \frac{\tau^{2n}}{n} \right)
\]

\[
= 1 + \frac{k}{2} + 2k [\tau - \log(1 - \tau^2)]. \tag{8.5}
\]

For \( \tau = 0 \), corresponding to a return at large distance, this checks with a known result (ref. 5, eq. (23)).

The real part of \( Z_m \) resulting from (8.5) is the d.c. resistance \( R_0 \) of (7.36), whereas the imaginary part defines the inductance

\[
L_0 = \frac{\mu}{4\pi} \left\{ \frac{1}{2} + 2[\tau - \log(1 - \tau^2)] \right\} \tag{8.6}
\]

\[
= \frac{\mu}{4\pi} \left\{ \frac{1}{2} - 2 \log 2 + 4\phi - 2e^{-2\phi} - 2 \log[\sinh(2\phi)] \right\}. \tag{8.7}
\]

The coefficient of \( \mu/4\pi \) in (8.6) is non-negative for \(-1 \leq \tau \leq 0\), for its derivative with respect to \( \tau \) vanishes only at \( \tau = 1 - \sqrt{2} \) in that interval, and at that point the coefficient has a positive minimum

\[
\frac{1}{2} + 2\{1 - \sqrt{2} - \log[2(\sqrt{2} - 1)]\} = 0.048.
\]

For small \( \phi \), (8.7) simplifies to

\[
L_0 = -\frac{\mu}{2\pi} \left[ \frac{3}{4} + \log \left( \frac{4c}{a} \right) \right]. \tag{8.8}
\]

The results (7.36) and (8.8) show that, for \( \phi = 0 \), because \( a = \infty \), one has \( R_0 = 0 \) and \( L_0 = \infty \) as in (5.18) and (5.19) under d.c. conditions, but the actual values are different for small \( \phi = c/a \). Now, the limit \( \phi = 0 \) can be considered to arise either from \( c = 0 \), finite \( a \), or from \( c \neq 0 \), \( a = \infty \). In the first case, the physical configuration with finite \( a \) is different from that with \( a = \infty \) treated in Sec. 5, so that there is no contradiction. In the second case, the approximation for small \( |k| \) of the present section is illegitimate, for the
value of (7.6) is not small for \( a = \infty \), even for small \( \omega \); the correct limiting process is then that described in the Appendix.

9. High-frequency asymptotic behaviour

For large \(|k|\), the Bessel function \( J_n = J_{n+k}(k) \) occurring in (7.34) is such that its order is almost equal to its argument. The first two terms of its asymptotic expansion are then (ref. 8, p. 247, eqs (4) and (7))

\[
J_{n+k}(k) = \frac{3^{1/2}}{6\pi} \left[ \frac{\Gamma(\frac{1}{3})}{(k/6)^{1/3}} - n \frac{\Gamma(\frac{2}{3})}{(k/6)^{2/3}} + O(k^{-4/3}) \right].
\] (9.1)

By setting \( n = 0 \) and \( n = 1 \) in (9.1), it follows that the asymptotic value of (7.35) is

\[
J'_k(k) = \frac{3^{1/6} \Gamma(\frac{2}{3})}{2^{1/3} \pi} k^{-2/3} + O(k^{-4/3}).
\] (9.2)

We thus have

\[
\frac{J_{n+k}(k)}{J_k(k)} = qk^{1/3} - n + O(k^{-1/3}),
\] (9.3)

where

\[
q = \frac{\Gamma(\frac{1}{3})}{2^{1/3} \pi^{1/3} \Gamma(\frac{2}{3})} = 1.088736 \ldots
\] (9.4)

On the other hand, the relation (7.32) shows that one has \( D_s = 1 + O(k^{-1}) \) for large \( |k| \). The first two terms of the asymptotic expansion of the matrix (7.34) resulting formally from (9.3) are thus

\[
M = qk^{1/3}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 2 & 3 & \ldots \\
1 & 1 & 2 & 3 & \ldots \\
2 & 2 & 2 & 3 & \ldots \\
3 & 3 & 3 & 3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

(9.5)
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It is emphasized that eq. (9.5) is not rigorously justified because the asymptotic expansion (9.3) is not uniform with respect to the integer $n$; neither is the limit result $D_0 = 1 + O(k^{-1})$ for $|k| \to \infty$ uniformly valid with respect to the integer $s$. Therefore the asymptotic expression (9.5) is correct for the entries $M_{ns}$ with $n, s \ll |k|$ only, and is certainly not uniformly valid in $n$ and $s$. Pursuing, however, the formal computation, we designate the matrices in (9.5) by $P$ and $Q$ and rewrite (9.5) as

$$M = q k^{1/3} Q - P.$$  \tag{9.6}

We next compute $t'Mt$ of eq. (7.40). We immediately have

$$t'Q t = \left( \frac{1}{2} + \tau + \tau^2 + \cdots \right)^2 = \left( \frac{1}{2} + \frac{\tau}{1 - \tau} \right)^2 = \frac{1}{4} \left( 1 + \frac{\tau}{1 - \tau} \right)^2.$$

\[ \tag{9.7}

On the other hand, we observe from (9.5) that

$$P_{ns} = \max(n, s), \quad (n, s = 0, 1, 2, \ldots).$$ \tag{9.8}

Hence

$$t'Pt = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} e_n e_s \max(n, s) \tau^{n+s} = \sum_{n=1}^{\infty} S_n \tau^n$$ \tag{9.9}

where

$$S_n = \sum_{m=0}^{n} e_m e_{n-m} \max(m, n - m).$$

To evaluate $S_n$, we distinguish the two cases of even or odd $n$:

$$S_{2n} = 2n + 2[(2n - 1) + (2n - 2) + \cdots + (n + 1)] + n = 3n^2, \quad (n \geq 1),$$ \tag{9.10}

$$S_{2n+1} = 2n + 1 + 2[2n + (2n - 1) + \cdots + (n + 1)]$$

$$= 3n^2 + 3n + 1 = (n + 1)^3 - n^3, \quad (n \geq 0).$$ \tag{9.11}
Inserting (9.10) and (9.11) into (9.9) we obtain

\[ t'Pt = 3 \sum_{n=1}^{\infty} n^2 \tau^{2n} + \tau^{-1}(1 - \tau^2) \sum_{n=1}^{\infty} n^3 \tau^{2n} \]

\[ = \frac{3\tau^2(1 + \tau^2)}{(1 - \tau^2)^3} + \tau^{-1}(1 - \tau^2) \frac{\tau^2(1 + 4\tau^2 + \tau^4)}{(1 - \tau^2)^4} \]

\[ = \frac{\tau(1 + \tau + \tau^2)}{(1 - \tau)^3(1 + \tau)}. \quad (9.12) \]

The derivation in (9.12) used the auxiliary results

\[ \sum_{n=1}^{\infty} n^2 \tau^n = \frac{\tau(1 + \tau)}{(1 - \tau)^3}; \quad \sum_{n=1}^{\infty} n^3 \tau^n = \frac{\tau(1 + 4\tau + \tau^2)}{(1 - \tau)^4} \]

deduced by differentiation of the geometric series \( \sum \tau^n \). By use of (9.6), (9.7) and (9.12) in (7.40), the asymptotic impedance is found to be

\[ \frac{Z_m}{R_0} = qk^{1/3}\left(\frac{1 + \tau}{1 - \tau}\right)^2 - \frac{4\tau(1 + \tau + \tau^2)}{(1 - \tau)^3(1 + \tau)}. \quad (9.13) \]

For a wide strip \((a \gg c)\), one has \( \phi = c/a \) by (2.3), and (6.7) gives \( \tau = -1 + 2\phi = -1 + 2c/a \). To the first order in \( c/a \), the impedance (9.13) then becomes

\[ \frac{Z_m}{R_0} = qk^{1/3}\left(\frac{c}{a}\right)^2 + \frac{a}{4c}. \quad (9.14) \]

Using (7.36), \( Z_m \) thus tends to \( R_\infty \) of (4.15) for \( a \to \infty \).

For a return at large distance \((\tau = 0)\), expression (9.13) reduces to the known result \( qk^{1/3} \) (ref. 5, eq. (24)).

From (7.15) and (9.5) we may formally derive that

\[ F_n = qk^{1/3} \sum_{s=0}^{\infty} \varepsilon_s \tau^s - \sum_{s=0}^{\infty} \varepsilon_s \max(n, s) \tau^s \]

\[ = \frac{1}{2} qk^{1/3} \frac{1 + \tau}{1 - \tau} - \frac{1}{2} n \frac{1 + \tau}{1 - \tau} - \frac{\tau^{n+1}}{(1 - \tau)^2} \quad (9.15) \]

for large \(|k|\). Hence, one has \( F_n/k = O(k^{-2/3}) \) in (7.42) so that, in the limit
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as \(|k| \to \infty\), this expression formally reduces to

\[ j(\theta) = \frac{I}{\pi a \sin \theta} \frac{\sinh(2\phi)}{\cosh(2\phi) + \cos(2\theta)} \]  

(9.16)

which is identical to the asymptotic linear current density in (4.5). Once again it is emphasized that the present derivation is formal and not rigorous, because the asymptotic expression (9.15) for \(F_n\) is not uniformly valid in \(n\). The derivation clearly fails at the edges \((\theta = 0\) and \(\theta = \pi)\) of the strip: the form of (7.41) shows that \(j(\theta)\) vanishes at the edges for all finite \(k\), whereas eq. (9.16) is infinite at the edges. The same difficulty already arises in the expressions (7.45) and (7.46) for the current density in the case of a return at large distance and was overlooked in a previous paper (ref. 5, eqs (19) and (20)). For that case, the convergence as \(|k| \to \infty\) of (7.45) to (4.9) is discussed in a separate paper where it is shown that expression (7.45) tends to

\[ j(\theta) = \frac{I}{\pi a \sin \theta}, \]  

(9.17)

equivalent to (4.9), only on the open interval \(0 < \theta < \pi\) and not uniformly. At the edges, \(j(\theta)\) keeps the value 0. Near \(\theta = 0\), \(j(\theta)\) increases steeply to (9.17) but that asymptotic expression fails in a region \(0 \leq \theta \leq C|k|^{-1/3}\) where \(C\) is some constant. The behaviour near \(\theta = \pi\) follows by symmetry.

REFERENCES


Appendix: the infinitely wide strip as the limit of the elliptic cylinder

As mentioned at the beginning of Sec. 7, we show that the solutions for the elliptic strip obtained in Sec. 7 tend for \(a \to \infty\) to the solutions for the infinitely wide strip of constant thickness (7.5) obtained in Sec. 5.
From $x = a \cos \theta$ ($0 \leq \theta \leq \pi$), one has $\theta = \pi/2 - \arcsin(x/a)$ which for large $a$ is approximated by $\theta = \pi/2 - x/a$ within the first order; hence

$$\cos(2n\theta) = (-1)^n \cos\left(\frac{2nx}{a}\right). \quad (A1)$$

By inserting (A1) into (6.15) with $h$ replaced by $2b \sin \theta \approx 2b = h$ and by using the notation $p$ from (5.10), the expression for the current density becomes

$$j(x) = -\frac{pl}{2\pi c} \left[ B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{2nx}{a}\right) \right] \quad (A2)$$

where we have set

$$B_n = (-1)^n A_n. \quad (A3)$$

Comparing the different frequency normalizations of (5.10) and (7.6), one obtains using (7.5)

$$k = \frac{pa}{8c}, \quad (A4)$$

so that the recurrence relations (7.9) and (7.10) become, using (6.7),

$$B_0 + B_1 = -\frac{8c}{pa}, \quad (A5)$$

$$B_{n-1} + 2B_n + B_{n+1} = -\frac{16c}{pa} \left( nB_n + e^{-2n\phi} \right), \quad (n = 1, 2, \ldots). \quad (A6)$$

We introduce the continuous variable

$$\alpha = \frac{n}{a} \quad (A7)$$

and the continuous function

$$B_n = -\frac{4c}{a} B\left(\frac{n}{a}\right) = -\frac{4c}{a} B(\alpha). \quad (A8)$$
Using (A7) and (A8), the recurrence relations (A5) and (A6) become

\[ B(0) + B\left(\frac{1}{a}\right) = \frac{2}{p}, \]  
\[ B\left(\alpha - \frac{1}{a}\right) + 2B(\alpha) + B\left(\alpha + \frac{1}{a}\right) = -\frac{16c}{p} \alpha B(\alpha) + \frac{4}{p} e^{-2n\phi}. \]  
\[ (A9) \]
\[ (A10) \]

For large \( a \), the factor \( e^{-2n\phi} \) becomes \( e^{-2na/a} \) using (2.3), and hence \( e^{-2ac} \) using (A7). Thus, for \( a \to \infty \), equations (A9) and (A10) tend to

\[ B(0) = \frac{1}{p}, \]
\[ 4B(\alpha) = -\frac{16c}{p} \alpha B(\alpha) + \frac{4}{p} e^{-2ac}, \]

with the solution

\[ B(\alpha) = \frac{e^{-2ac}}{p + 4ac}. \]  
\[ (A11) \]

By inserting (A8) into (A2) we obtain

\[ j(x) = \frac{2pI}{\pi a} \left[ \frac{1}{2}B(0) + \sum_{n=1}^{\infty} B\left(\frac{n}{a}\right) \cos\left(\frac{2nx}{a}\right) \right]. \]  
\[ (A12) \]

The Fourier series of (A12) tends to a Fourier integral as \( a \to \infty \), as can be rigorously justified by means of the Euler–Maclaurin summation formula \( A1 \) yielding

\[ \lim_{a \to \infty} \frac{1}{a} \left[ \frac{1}{2}B(0) + \sum_{n=1}^{\infty} B\left(\frac{n}{a}\right) \cos\left(\frac{2nx}{a}\right) \right] = \int_{0}^{\infty} B(\alpha) \cos(2\alpha x) d\alpha. \]  
\[ (A13) \]

Using (A11) and (A13) and changing \( \alpha \) into \( \alpha/2 \), it follows that (A12) tends to (5.20) as \( a \to \infty \).
Using (6.7), (A3) and (A8), the impedance (6.14) becomes

\[
Z_m = -\frac{i\omega\mu}{2\pi} \left( \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n e^{-2n\phi} \right)
= \frac{2i\omega\mu c}{\pi a} \left[ \frac{1}{2}B(0) + \sum_{n=1}^{\infty} B\left(\frac{n}{a}\right) e^{-2n\phi/a} \right],
\]

where the fact that \( \phi = c/a \) for large \( a \) was used. Using (A13) with \( x = 0 \) and \( B(n/a) \) replaced by \( B(n/a) e^{-2nc\phi/a} \), and hence with \( B(\alpha) \) of (A11) replaced by \( B(\alpha) e^{-2\alpha c} \), the limit for \( a \to \infty \) becomes

\[
Z_m = \frac{2i\omega\mu c}{\pi} \int_0^{\infty} \frac{e^{-4\alpha c}}{p + 4\alpha c} \, d\alpha.
\]

Using (5.10) and changing \( \alpha \) into \( \alpha/2 \), this reduces to (5.9).

REFERENCES FOR APPENDIX


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