ON A NEW THREE-TERM RECURRENCE RELATION IN TOEPLITZ SYSTEM THEORY

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Abstract

This paper is concerned with a novel three-term recurrence relation for a well-defined family of symmetric or antisymmetric polynomials. In the context where it was originally discovered, this recurrence yields a method for checking whether a given polynomial is devoid of zeros in the closed unit disc. As shown in this paper, the same recurrence, used in reverse order, gives rise to a new algorithm for solving the classical linear prediction problem relative to a given positive definite Toeplitz matrix. This algorithm can be viewed as a split version of a suitable analogue of the well-known Levinson algorithm.

Keywords: Levinson algorithm, linear prediction, orthogonal polynomials, polynomial stability test, three-term recurrence, Toeplitz matrices

1. Introduction

Positive definite Toeplitz matrices are an important research subject in various branches of applied mathematics. In particular, in digital signal processing applications, they occur most naturally as autocorrelation matrices of sampled signal records. The problem of solving a system of linear equations that exhibits the positive definite Toeplitz structure plays a central role in the field. It occurs in a collection of filtering, modelling and estimation techniques. The reader is especially referred to a survey by the authors ¹) for an introduction to the subject and for a presentation of some recent developments.

The linear prediction problem for stationary signals is an important particular case of the problem alluded to above; it essentially consists in determining the first column of the inverse of a positive definite Toeplitz matrix. This problem is classically solved by means of the Levinson algorithm ²,³), which recursively computes the linear prediction vectors...
P. Delsarte and Y. Genin

associated with the sequence of nested Toeplitz submatrices of the given Toeplitz matrix. When interpreted in polynomial terms, the underlying recurrence relation has the structure of the well-known one-step recurrence formula, involving a single 'reflection coefficient', for a family of Szegö polynomials orthogonal on the unit circle of the complex plane \(^4\). In this interpretation, the so-called predictor polynomials, built from the linear prediction vectors, simply are the mirror images of the corresponding Szegö polynomials.

The Levinson algorithm can be viewed as a kind of 'dual' of the Schur–Cohn algorithm \(^5\), which is an efficient method for testing whether a given polynomial is stable, in the sense that it is devoid of zeros in the closed unit disc of the complex plane. In fact, the latter algorithm is based on the same one-step recurrence relation as the former. The Schur–Cohn method uses this recurrence from high degree to low degree and computes the reflection coefficient by simple polynomial evaluation (at the origin); the Levinson method uses the same basic recurrence from low degree to high degree and computes the reflection coefficient via an inner product formula that involves the current predictor coefficients and the entries of the Toeplitz matrix.

Lower complexity substitutes for the Schur–Cohn polynomial stability test and for the Levinson algorithm were presented by Bistritz \(^6,7\) and by the authors, under the name of the split Levinson algorithm \(^8,9\), respectively. As explained in refs. 1 and 10, there exists a 'duality' between the split Levinson algorithm and a variant of the Bistritz stability test, which is analogous to the aforementioned relationship between the classical Levinson algorithm and the Schur–Cohn stability test. In fact, both methods are suitable implementations of the same two-step recurrence relation for symmetric (or antisymmetric) polynomials, first discovered by Davis in a digital filtering context \(^11\). This recurrence can be interpreted, via a simple change of variable, as a standard three-term formula for polynomials orthogonal on the real line \(^4,8,12\).

In a recent contribution to the model reduction problem, Hwang and Hsieh investigated a novel two-step recurrence relation for symmetric polynomials \(^13\). It gives rise to some new losslessness test methods for rational functions and, hence, to new polynomial stability test algorithms \(^13,14\). The Hwang–Hsieh recurrence is quite different from the Davis recurrence (which underlies the Bistritz and split Levinson algorithms). However, it can also be interpreted as a three-term formula for orthogonal polynomials on the real line. This phenomenon is best explained from a unifying approach to both types of recurrences in the framework of lossless function theory \(^14\).

One of the main objectives of this paper is to show that the duality pattern
Three-term recurrence in Toeplitz system theory

mentioned above, concerning the Schur–Cohn test (vs. the Levinson algorithm) and the Bistritz test (vs. the split Levinson algorithm), extends to the case of the Hwang–Hsieh test. More precisely, we will describe a new efficient algorithm for solving the usual Toeplitz linear prediction problem, which is based on the two-step recurrence relation that underlies the Hwang–Hsieh test. Of course, this algorithm differs from its dual test method in the running order of the recurrence and in the computation of the coefficients.

There is another reason why the new linear prediction method described herein should not be viewed as an isolated item. In fact, this method can be described in the framework of the generalized Levinson algorithm \(^{15}\), which is closely related to the Dewilde–Dym technique in lossless inverse scattering \(^{16}\). The generalization in question is based on an order lowering ‘reduction operation’ for Toeplitz matrices, which involves a well-defined parameter \(\zeta\). (The classical Levinson algorithm corresponds to the choice \(\zeta = 0\).) The new algorithm discussed herein occurs as a ‘split version’ of a special case of the generalized Levinson algorithm, where the defining parameter \(\zeta\) is set equal to 1 or to \(-1\).

This approach shows that our new linear prediction method may be interpreted as a suitable ‘deflated implementation’ of the split Levinson algorithm. The deflation possibility relies on the fact that the order of the relevant Toeplitz matrix can be lowered by one unit when all singular predictor polynomials have a zero in common \(^{17,18}\).

Section 2 contains some material about the generalized Levinson algorithm, with special emphasis on the cases \(\zeta = 1\) and \(\zeta = -1\). Section 3 introduces the appropriate symmetric and antisymmetric predictor polynomials and discusses their connections with the usual predictor polynomials. Section 4 gives a derivation of two-step recurrence relations for both families of symmetric and antisymmetric polynomials and shows their equivalence with the Hwang–Hsieh formulas. Section 5 explains how the coefficients occurring in these recurrence relations can be computed in an efficient manner, which leads to a novel solution method for the linear prediction problem.

2. Background and preliminaries

To begin with, let us recall some material from ref. 15, in the special case of real data. Consider a real Toeplitz matrix \(C_k\) of order \(k \geq 1\), i.e. a matrix

Philips Journal of Research Vol. 45 No. 1 1990

69
Thus, the \((i, j)\) entry of \(C_k\) equals \(c_k(i - j)\), for \(0 \leq i, j \leq k\). Throughout the paper, we assume that \(C_k\) is symmetric, i.e. \(c_k(-i) = c_k(i)\) for all \(i\).

Let there be given a real number \(\zeta_k\) in the closed interval \(-1 \leq \zeta_k \leq 1\). From the data \(c_k(i)\) we construct the Toeplitz matrix \(C_{k-1}\), of order \(k\), by defining its \((i, j)\) entry \(c_{k-1}(i - j)\) via the formula

\[
c_{k-1}(i) = (1 + \zeta_k^2)c_k(i) - \zeta_k[c_k(i - 1) + c_k(i + 1)],
\]

for \(1 - k \leq i \leq k - 1\). In matrix form, this reads

\[
C_{k-1} = Z_k^T C_k Z_k,
\]

with the \((k + 1) \times k\) bidiagonal Toeplitz matrix

\[
Z_k = \begin{bmatrix}
-\zeta_k \\
1 & -\zeta_k \\
\vdots & \ddots & \ddots \\
& \cdots & 1
\end{bmatrix}.
\]

The \(k \times k\) Toeplitz matrix \(C_{k-1}\) will be referred to as the reduced form of \(C_k\) with respect to the point \(\zeta_k\). It is obvious from the definition that \(C_{k-1}\) is symmetric (since so is \(C_k\)). In what follows we suppose that \(C_k\) is positive definite. In view of relation (2.3), this implies that \(C_{k-1}\) also is positive definite.

Consider the predictor polynomial \(a_k(z) = \sum_{i=0}^{k} a_{k,i} z^i\) associated with the positive definite Toeplitz matrix \(C_k\). By definition, its coefficient vector \(a_k = (a_{k,0}, \ldots, a_{k,k})^T\) is the solution of the system of linear equations

\[
C_k a_k = (\sigma_k, 0, \ldots, 0)^T.
\]

We assume the usual normalization \(a_{k,0} = 1\); this uniquely defines the real
number $\sigma_k$ in relation (2.5). Note the property $\sigma_k = a_k^T C_k a_k$, whence $\sigma_k > 0$. The positive number $\sigma_k$ is called the prediction error squared norm relative to $C_k$.

Let $a_{k-1}(z)$ denote the predictor polynomial associated with the reduced form $C_{k-1}$ of $C_k$ (with respect to $\zeta_k$), and let $\sigma_{k-1}$ be the corresponding squared norm. As shown in ref. 15, the pair $(a_k(z), \sigma_k)$ is related to the pair $(a_{k-1}(z), \sigma_{k-1})$ through the formulas

$$a_k(z) = (1 - \mu_k \zeta_k z)a_{k-1}(z) + v_k z \hat{a}_{k-1}(z),$$

$$\sigma_k = \mu_k \sigma_{k-1},$$

where $\hat{a}_{k-1}(z) = z^{k-1} a_{k-1}(z^{-1})$ is the mirror image of $a_{k-1}(z)$. The coefficient $v_k$ in relation (2.6) can be expressed as follows:

$$v_k = \varepsilon_k (1 - \mu_k \zeta_k^2), \quad \text{with } \varepsilon_k = \hat{a}_k(\zeta_k)/a_k(\zeta_k).$$

This implies the identity $1 - \mu_k = \varepsilon_k v_k$. Since $a_k(z)$ is devoid of zeros in the closed unit disc $|z| \leq 1$, we have the inequalities $-1 \leq \varepsilon_k \leq 1$. More precisely, we have $-1 < \varepsilon_k < 1$ if $-1 < \zeta_k < 1$, in contrast with $\varepsilon_k = 1$ if $\zeta_k = 1$ and $\varepsilon_k = (-1)^k$ if $\zeta_k = -1$. The parameter $\mu_k$ satisfies

$$0 < \mu_k \leq 1,$$

with equality $\mu_k = 1$ if and only if $\varepsilon_k = 0$. In fact, this approach leads to a new positivity criterion for Toeplitz matrices; it can be shown that $C_k$ is positive definite if and only if $\mu_k$ obeys the constraint (2.9) and $C_{k-1}$ is positive definite.

Consider the family of positive definite Toeplitz matrices $C_0, C_1, \ldots, C_n$ (of order $1, 2, \ldots, n + 1$), the entries of which are related by formula (2.2), for a given sequence of real numbers $\zeta_1, \zeta_2, \ldots, \zeta_n$ (satisfying $|\zeta_k| \leq 1$). Relations (2.6) and (2.7) can be used to determine recursively the predictor polynomial $a_k(z)$ and the squared norm $\sigma_k$ associated with $C_k$, for $k = 0, 1, \ldots, n$. A specific computation scheme of that type is given in ref. 15, under the name of the generalized Levinson algorithm. In the particular case $\zeta_k = 0$, the reduced form $C_{k-1}$ of $C_k$ is nothing but its upper (or lower) Toeplitz submatrix, and formula (2.6) is the well-known recurrence relation that underlies the standard Levinson algorithm.

Two other values of $\zeta_k$ deserve special attention in the theory, namely $\zeta_k = 1$ and $\zeta_k = -1$. As we will see in Sec. 5, there exist interesting ‘split versions’ of the generalized Levinson algorithm in these cases. (This phenomenon is
P. Delsarte and Y. Genin

analogous to that observed in the classical case \( \zeta_k = 0 \), for which the split Levinson algorithm\(^8,9\) can be used as an economical substitute for the standard Levinson method.) In what follows we concentrate exclusively on either of the particular situations \( \zeta_k = 1 \) (for all \( k \)) or \( \zeta_k = -1 \) (for all \( k \)).

In the first case, \( \zeta_k = 1 \), we have \( t_k = 1 \). Hence, the recurrence relation (2.6) can be written as follows:

\[
a_k(z) = a_{k-1}(z) + z \hat{a}_{k-1}(z) - \mu_k z [a_{k-1}(z) + \hat{a}_{k-1}(z)].
\]  

Combining formula (2.10) with its mirror image, in two different ways, we obtain the two remarkable identities

\[ a_k(z) - z \hat{a}_k(z) = (1 - z) [a_{k-1}(z) + z \hat{a}_{k-1}(z)], \quad (2.11) \]

\[ a_k(z) - \hat{a}_k(z) = \mu_k (1 - z) [a_{k-1}(z) + \hat{a}_{k-1}(z)]. \quad (2.12) \]

They relate some antisymmetric polynomials, in the left-hand sides, to some symmetric polynomials, inside square brackets in the right-hand sides. Recall that a polynomial \( x(z) \) is said to be symmetric or antisymmetric if it satisfies \( x(z) = x(z) \) or \( x(z) = -x(z) \) respectively.

The second case, \( \zeta_k = -1 \), yields \( t_k = (-1)^k \). Instead of relation (2.10) we obtain

\[ a_k(z) = a_{k-1}(z) + (-1)^k z \hat{a}_{k-1}(z) + \mu_k z [a_{k-1}(z) - (-1)^k \hat{a}_{k-1}(z)], \quad (2.13) \]

from which we deduce the symmetric–antisymmetric identities

\[ a_k(z) + (-1)^k z \hat{a}_k(z) = (1 + z) [a_{k-1}(z) + (-1)^k z \hat{a}_{k-1}(z)], \quad (2.14) \]

\[ a_k(z) - (-1)^k \hat{a}_k(z) = \mu_k (1 + z) [a_{k-1}(z) - (-1)^k \hat{a}_{k-1}(z)]. \quad (2.15) \]

3. Symmetric and antisymmetric predictors

Let us first consider the case \( \zeta_k = 1 \). Depending on the parity of \( k \) we define the symmetric predictor polynomial \( s_k(z) \), for \( k = 0, 1, \ldots, n \), as follows:

\[
s_{2j}(z) = a_{2j}(z) + z \hat{a}_{2j}(z),
\]

\[
s_{2j+1}(z) = \sigma_{2j+1}^{-1} [a_{2j+1}(z) + \hat{a}_{2j+1}(z)].
\]  

In particular, \( s_0(z) = 1 + z \) and \( s_1(z) = [c_1(0) + c_1(1)]^{-1}(1 + z) \). Note that \( s_{2j}(z) \)
and $s_{2j+1}(z)$ both have degree $2j+1$, and that they vanish at the point $z = -1$. Note also the normalization $s_{2j}(0) = 1$. In view of formulas (2.11) and (2.12), we can alternatively define the same polynomials $s_k(z)$ by

$$s_{2j}(z) = (1 - z)^{-1}[a_{2j+1}(z) - za_{2j+1}(z)],$$

$$s_{2j+1}(z) = s_{2j+2}(1 - z)^{-1}[a_{2j+2}(z) - a_{2j+2}(z)].$$

Therefore, by suitable elimination of $a_k(z)$, we can express the usual predictor $a_k(z)$ in terms of the symmetric predictors $s_k(z)$ and $s_{k-1}(z)$; we obtain

$$(1 + z)a_{2j}(z) = s_{2j}(z) + a_{2j}z(1 - z)s_{2j-1}(z),$$

$$(1 + z)a_{2j+1}(z) = a_{2j+1}zs_{2j+1}(z) + (1 - z)s_{2j}(z).$$

By use of the polynomial relations (3.1) and (3.2) we can also derive interesting identities for some rational functions involving predictor polynomials, namely

$$\frac{2s_{2j}(z)}{(1 - z^2)s_{2j-1}(z)} = \sigma_{2j}\left[\frac{1 - z}{1 + z} + \frac{a_{2j}(z) + a_{2j}(z)}{a_{2j}(z) - a_{2j}(z)}\right],$$

$$\frac{2(1 - z)s_{2j}(z)}{(1 + z)s_{2j+1}(z)} = \sigma_{2j+1}\left[\frac{1 - z}{1 + z} + \frac{a_{2j+1}(z) - a_{2j+1}(z)}{a_{2j+1}(z) + a_{2j+1}(z)}\right].$$

From the fact that $a_k(z)$ is devoid of zeros in the closed unit disc, it follows that the right-hand sides of formulas (3.4) and (3.5) are lossless functions of degrees $2j$ and $2j+1$ respectively. (Recall that a rational function $f(z)$, with real coefficients, is said to be lossless if it satisfies the identity $f(z) + f(z^{-1}) = 0$ and if the real part of $f(z)$ is non-negative in the unit disc $|z| < 1$.) This implies that the zeros of the symmetric predictors $s_k(z)$ are simple and are located on the unit circle $|z| = 1$; moreover, the zeros of $s_{2j}(z)/(1 + z)$ alternate with those of $(1 - z)s_{2j-1}(z)$, and the zeros of $s_{2j+1}(z)$ alternate with those of $(1 - z)s_{2j}(z)/(1 + z)$.

Next, let us examine the case $\zeta_k = -1$. Here we introduce the antisymmetric predictor polynomials $t_k(z)$, for $k = 0, 1, \ldots, n$. Depending on the parity of $k$, the definition is the following:

$$t_{2j}(z) = a_{2j}(z) - za_{2j}(z),$$

$$t_{2j+1}(z) = s_{2j+1}^{-1}[a_{2j+1}(z) - a_{2j+1}(z)].$$
In particular, \( t_0(z) = 1 - z \) and \( t_1(z) = [c_1(0) - c_1(1)]^{-1}(1 - z) \). Note that \( t_{2j}(z) \) and \( t_{2j+1}(z) \) have degree \( 2j + 1 \), that they vanish at the point \( z = 1 \), and that \( t_{2j}(0) \) equals 1. By use of formulas (2.14) and (2.15) we obtain the equivalent definition

\[
t_{2j}(z) = (1 + z)^{-1} [a_{2j+1}(z) - z \hat{a}_{2j+1}(z)],
\]

\[
t_{2j+1}(z) = \sigma_{2j+2}(1 + z)^{-1} [a_{2j+2}(z) - \hat{a}_{2j+2}(z)].
\]

By combining the relations (3.6) and (3.7) we can deduce an expression for \( a_k(z) \) in terms of \( t_k(z) \) and \( t_{k-1}(z) \); the result is

\[
(1 - z)a_{2j}(z) = t_{2j}(z) - \sigma_{2j} z(1 + z)t_{2j-1}(z),
\]

\[
(1 - z)a_{2j+1}(z) = -\sigma_{2j+1} z t_{2j+1}(z) + (1 + z)t_{2j}(z).
\]

Furthermore, instead of formulas (3.4) and (3.5), we have the following rational function identities:

\[
\frac{2t_{2j}(z)}{(1 - z^2)t_{2j-1}(z)} = \sigma_{2j} \left[ \frac{1 + z}{1 - z} \left( \frac{1 + z}{a_{2j}(z)} \right) \right],
\]

\[
\frac{2(1 + z)t_{2j}(z)}{(1 - z)t_{2j+1}(z)} = \sigma_{2j+1} \left[ \frac{1 + z}{1 - z} \left( \frac{1 + z}{a_{2j+1}(z)} \right) \right].
\]

As a consequence, the zeros of the antisymmetric predictors \( t_k(z) \) are simple and are located on the unit circle; moreover, the zeros of \( t_{2j}(z)/(1 - z) \) alternate with those of \( (1 + z)t_{2j-1}(z) \), and the zeros of \( t_{2j+1}(z) \) alternate with those of \( (1 + z)t_{2j}(z)/(1 - z) \).

For future use, let us introduce the normalized reduced versions, \( \tilde{s}_k(z) \) and \( \tilde{t}_k(z) \), of the symmetric and antisymmetric predictors \( s_k(z) \) and \( t_k(z) \). The definitions are

\[
\tilde{s}_k(z) = \frac{s_k(z)}{2^k(1 + z)}, \quad \tilde{t}_k(z) = \frac{t_k(z)}{2^k(1 - z)},
\]

for \( k = 0, 1, \ldots, n \). Both polynomials \( \tilde{s}_k(z) \) and \( \tilde{t}_k(z) \) are symmetric. Their degree properties are as follows:

\[
\deg \tilde{s}_{2j}(z) = \deg \tilde{t}_{2j}(z) = \deg \tilde{s}_{2j+1}(z) = \deg \tilde{t}_{2j+1}(z) = 2j.
\]
Three-term recurrence in Toeplitz system theory

Remark

The identities involving the antisymmetric predictors $t_k(z)$ are seen to be quite similar to those involving the symmetric predictors $s_k(z)$. Roughly speaking, they can be transformed into each other via the change of variable $z \rightarrow -z$. However, it is important to note that the relationship in question is only formal; it admits no direct ‘numerical interpretation’ since the predictor polynomial $a_k(z)$ and the squared norm $\sigma_k$ relative to the symmetric case ($\zeta_k = 1$) are generally different from those relative to the antisymmetric case ($\zeta_k = -1$), for the same Toeplitz matrix $C_n$.

4. Recurrence relations

This section plays a central role in the paper. It is devoted to deriving and discussing some remarkable recurrence formulas for both systems of symmetric and antisymmetric predictor polynomials $s_k(z)$ and $t_k(z)$ relative to a given Toeplitz matrix $C_n$. Let us emphasize that the recurrences in question are radically different from those underlying the split Levinson algorithm $^8$, which also involve some ‘symmetric or antisymmetric predictors’ (not the same as here).

In the symmetric case ($\zeta_k = 1$), the key idea is to rewrite formula (2.10) in terms of the polynomials $s_k(z)$ with the help of the identities (3.3). By elementary manipulation we thus obtain the three-term (two-step) recurrence relation

$$s_{2j}(z) = 2\beta_{2j} z s_{2j-1}(z) + (1 - z)^2 s_{2j-2}(z),$$

$$s_{2j+1}(z) = 2\beta_{2j+1} s_{2j}(z) + (1 - z)^2 s_{2j-1}(z),$$

where the coefficient $\beta_k$ is given by

$$\beta_{2j+1} = \sigma_{2j+1} - \sigma_{2j-1}, \quad \beta_{2j} = \sigma_{2j-1} - \sigma_{2j}.$$  \hspace{1cm} (4.2)

Note that $\beta_k$ is positive, because of the constraint (2.9).

Equivalently, the recurrence (4.1) can be written in terms of the normalized reduced polynomials $\bar{s}_k(z)$, defined by the first formula (3.11); the result is the following:

$$\bar{s}_{2j}(z) = \beta_{2j} z \bar{s}_{2j-1}(z) + \frac{1}{2} (1 - z)^2 \bar{s}_{2j-2}(z),$$

$$\bar{s}_{2j+1}(z) = \beta_{2j+1} \bar{s}_{2j}(z) + \frac{1}{2} (1 - z)^2 \bar{s}_{2j-1}(z).$$

Recurrence relations of this unusual form were recently considered by Hwang.
P. Delsarte and Y. Genin

and Hsieh in the context of the model reduction problem\textsuperscript{13}), without any reference to the linear prediction problem for Toeplitz matrices. They can be used to check whether a given rational function is lossless or, equivalently, whether a given polynomial is devoid of zeros in the closed unit disc\textsuperscript{13,14}).

Let us briefly explain the losslessness test in question. Consider a rational function $f_n(z)$ of positive degree $n$, with real coefficients, enjoying the property $f_n(z) + f_n(z^{-1}) = 0$. Without loss of generality, assume $f_n(z)$ to have a pole at the point $z = -1$. (Otherwise, replace $f_n(z)$ by $1/f_n(z)$.) Then, depending on the parity of $n$, we can write $f_n(z)$ in the form

$$f_{2r}(z) = \frac{4\tilde{s}_{2r}(z)}{(1 - z^2)\tilde{s}_{2r-1}(z)}, \quad f_{2r+1}(z) = \frac{(1 - z)\tilde{s}_{2r}(z)}{(1 + z)\tilde{s}_{2r+1}(z)},$$

for some well-defined symmetric polynomials $\tilde{s}_n(z)$ and $\tilde{s}_{n-1}(z)$, enjoying the degree property (3.12), with the normalization $\tilde{s}_{2r}(0) = 4^{-r}$. Thus, the first and second expressions (4.4) formally coincide with the left-hand sides of formulas (3.4) and (3.5) respectively, where $j$ is replaced by $r$.

Using the polynomials $\tilde{s}_n(z)$ and $\tilde{s}_{n-1}(z)$ thus defined as initial conditions for the descending version of the recurrence (4.3), we can construct a sequence of symmetric polynomials $\tilde{s}_k(z)$, for $k = n - 2, n - 3, \ldots, 0$, together with a sequence of real numbers $\beta_n, \beta_{n-1}, \ldots, \beta_1$, given by

$$\beta_k = \frac{\tilde{s}_k(1)}{\tilde{s}_{k-1}(1)}. \quad (4.5)$$

Next, let us define the real numbers $\lambda_k$ as follows:

$$\lambda_k = \frac{\tilde{s}_k(-1)}{\tilde{s}_{k-1}(-1)}, \quad (4.6)$$

for $k = 1, 2, \ldots, n$. In view of formulas (4.3), these numbers can be computed from the $\beta_k$ coefficients by means of the recurrence

$$\lambda_k = \lambda_{k-1}^{-1} + (-1)^{k-1}\beta_k, \quad (4.7)$$

with the initial condition $\lambda_0 = \infty$. The result alluded to above (proved in ref. 14) states that $f_n(z)$ is a lossless function if and only if the $\lambda_k$ parameters are well defined and satisfy the inequality system

$$0 < \lambda_1 < \lambda_2^{-1} < \lambda_3 < \cdots < \lambda_{n-1}^{-1} < 1. \quad (4.8)$$
Note that this merely is a compact way of writing the 2n inequalities $\beta_k > 0$ and $\lambda_k > 0$ for $k = 1, 2, \ldots, n$.

It is interesting to interpret the criterion (4.8) in the framework of Toeplitz matrices. From the polynomials $\tilde{s}_n(z)$ and $\tilde{s}_{n-1}(z)$ we can construct a 'predictor polynomial' $a_n(z)$ by use of formulas (3.11) and (3.3). The appropriate coefficient $\sigma_n$ is obtained in terms of the parameters $\beta_k$ by means of the recurrence (4.2), initialized with an arbitrary positive number $\sigma_0$. Then, provided that the polynomials $a_n(z)$ and $\delta_n(z)$ are coprime, there exists a unique symmetric Toeplitz matrix $C_n$ satisfying eq. (2.5), with $k$ replaced by $n$. It can be shown that $f_n(z)$ is lossless if and only if $C_n$ is positive definite. To prove this statement, we simply have to verify that the losslessness criterion (4.8) is equivalent to the Toeplitz positive definiteness criterion (2.9), i.e. in the present situation, to the inequality system

$$\sigma_0 > \sigma_1 > \cdots > \sigma_n > 0.$$  \hspace{1cm} (4.9)

We argue as follows. Define the real numbers $\phi_0, \phi_1, \ldots, \phi_n$ by

$$\phi_{2j} = \sigma_{2j}, \quad \phi_{2j+1} = \sigma_{2j+1}^{-1}. \hspace{1cm} (4.10)$$

From relations (4.2) and (4.7) we deduce the identity

$$\lambda_k - \lambda_{k-1}^{-1} = \phi_k - \phi_{k-1}^{-1}. \hspace{1cm} (4.11)$$

Since $\lambda_0 = \infty$, this immediately shows that condition (4.9) is equivalent to condition (4.8) supplemented with $0 < \sigma_0 < \infty$.

Note that $\lambda_{2j}$ and $\lambda_{2j+1}^{-1}$ are equal to the half residues of the functions $f_{2j}(z)$ and $f_{2j+1}(z)$ in the left-hand sides of equations (3.4) and (3.5) at the pole $z = -1$. This yields a clear interpretation of the inequalities $\lambda_{2j} > \sigma_{2j}$ and $\lambda_{2j+1}^{-1} > \sigma_{2j+1}$ that readily follow from relation (4.11), by induction.

Very similar results can be obtained in the antisymmetric case ($\zeta_k = -1$). Let us summarize them, without going into all details. Inserting the expressions of $a_k(z)$ and $a_{k-1}(z)$ given by formulas (3.8) into the identity (2.13), we deduce the three-term recurrence relation

$$t_{2j}(z) = -2\beta_{2j}zt_{2j-1}(z) + (1 + z)^2t_{2j-2}(z),$$
$$t_{2j+1}(z) = 2\beta_{2j+1}t_{2j}(z) + (1 + z)^2t_{2j-1}(z), \hspace{1cm} (4.12)$$

where the coefficient $\beta_k$ is determined by the same formula (4.2) as in the symmetric case. (Let us stress that, for a given Toeplitz matrix $C_n$, the numerical
values of the parameters $\beta_k$ in the recurrences (4.1) and (4.12) are generally not the same, for the reason explained at the end of Sec. 3.)

When written in terms of the symmetric polynomials $\tilde{t}_k(z)$, which are the normalized reduced images of the antisymmetric predictors $t_k(z)$, the recurrence (4.12) becomes

$$
\tilde{t}_{2j}(z) = -\beta_{2j}z\tilde{t}_{2j-1}(z) + \frac{1}{4}(1+z)^2\tilde{t}_{2j-2}(z),
\tilde{t}_{2j+1}(z) = \beta_{2j+1}\tilde{t}_{2j}(z) + \frac{1}{4}(1+z)^2\tilde{t}_{2j-1}(z).
$$

(4.13)

This is an alternative version of the Hwang–Hsieh formula \(^{13}\)). It yields a losslessness criterion which is equivalent, from a theoretical viewpoint, to the result explained above (in the case $\zeta_k = 1$). Further details about this subject can be found in ref. 14 (where the notations $q_+^*$ and $q_-^*$ are used instead of $\tilde{s}_k$ and $\tilde{t}_k$). Note that the coefficient $\beta_k$ of the recurrence relation (4.13) can be determined as the ratio

$$
\beta_k = \frac{\tilde{t}_k(-1)}{\tilde{t}_{k-1}(-1)}.
$$

(4.14)

The $\lambda_k$ numbers involved in the losslessness criterion (4.8), relative to the case $\zeta_k = -1$, can be computed from the $\beta_k$ numbers by means of the recurrence (4.7), with $\lambda_0 = \infty$. In explicit terms, they are given by

$$
\lambda_k = \frac{\tilde{t}_k(1)}{\tilde{t}_{k-1}(1)}.
$$

(4.15)

Note that the roles of the distinguished points 1 and $-1$ in formulas (4.5), (4.6) and (4.14), (4.15) are interchanged.

5. Split versions of the generalized Levinson algorithm (in the case $\zeta_k = \pm 1$)

We will now see how either of the recurrence relations (4.1) or (4.12), for the symmetric or antisymmetric predictor polynomials $s_k(z)$ or $t_k(z)$, can be used to compute the ordinary predictor polynomial $a_n(z)$ and the corresponding squared norm $\sigma_n$ associated with a given positive definite real Toeplitz matrix $C_n$. Thus we will obtain two linear prediction algorithms which can be interpreted as 'split versions' of the generalized Levinson algorithm described in ref. 15 (in the case $\zeta_k = \pm 1$). These new algorithms are considerably more economical than their 'non-split' counterparts. In fact, their complexity is comparable with that of the split Levinson algorithm \(^{8,9}\)); they involve about the same number of multiplications as it does, but they require a larger number of additions. (In the context of the linear prediction problem,
the name 'split algorithm' refers to a recursive method that processes suitable symmetric or antisymmetric predictors, instead of the usual unstructured predictors.)

Let us first treat the symmetric case ($e_k = 1$) in detail. We mainly have to explain how to compute the coefficients $\beta_k$ that occur in recurrence (4.1). To that end, let us introduce both symmetric polynomials

$$u_k(z) = \frac{a_k(z) - z \hat{a}_k(z)}{1 - z}, \quad v_k(z) = \frac{a_k(z) - \hat{a}_k(z)}{1 - z},$$

(5.1)

of degree $k$ and $k - 1$ respectively. (Thus, $s_{2j}(z)$ is equal to $u_{2j+1}(z)$, and $s_{2j+1}(z)$ is proportional to $v_{2j+2}(z)$; see relations (3.2).) As explained below, the $(k + 1)$-dimensional coefficient vectors $u_k$ and $v_k$ of the polynomials $u_k(z)$ and $v_k(z)$ satisfy the Toeplitz linear systems

$$C_k u_k = \sigma_k [a_k(1)]^{-1} (1, 1, \ldots, 1)^T,$$  

(5.2)

$$C_k (v_k - u_k) = (0, 0, \ldots, 0, -\sigma_k)^T.$$  

(5.3)

The relation (5.2) can be proved as follows. Let $C'_{k+1}$ denote any symmetric Toeplitz extension of order $k + 2$ of the Toeplitz matrix $C_k$. In view of definition (2.5), we can write an identity of the form

$$C'_{k+1} (a_k - z \hat{a}_k) = (\gamma_k, 0, \ldots, 0, -\gamma_k)^T,$$  

(5.4)

where $a_k$ and $z \hat{a}_k$ denote the $(k + 2)$-dimensional coefficient vectors of $a_k(z)$ and $z \hat{a}_k(z)$. Since we have $a_k - z \hat{a}_k = (1 - z) u_k$, we immediately deduce from eq. (5.4) that all components of the vector $C_k u_k$ have the same value. To determine this value, and thus to complete the proof, it suffices to use the identity $a_k^T C_k u_k = \sigma_k$. As to the companion result (5.3), it is an immediate consequence of equation (2.5), for we have $v_k = u_k - \hat{a}_k$ by definition (5.1).

Next, we introduce some real numbers $\omega_k$, for $k = 1, 2, \ldots, n$; they are defined by means of the inner product formula

$$\omega_k = \sum_{i=0}^{k} c_k(i) s_{k-1,i},$$  

(5.5)

where $s_{k-1,i}$ denotes the coefficient of $z^i$ in the symmetric predictor $s_{k-1}(z)$. Thus, $\omega_k$ is the first component of the vector $C_k s_{k-1}$. By use of formulas (3.2),
(5.1), (5.2) and (5.3), we readily obtain the expressions

\[ \omega_{2j} = 1/a_{2j}(1), \quad \omega_{2j+1} = \sigma_{2j+1}/a_{2j+1}(1). \tag{5.6} \]

Equivalently, \( \omega_k = 2/s_k(1) = 1/2^k s_k(1) \), by definitions (3.1) and (3.11). As an immediate consequence, the expression (4.5) for the recurrence coefficient \( \beta_k \) can be written in the remarkable form

\[ \beta_k = \omega_{k-1}/2\omega_k, \tag{5.7} \]

for \( k = 1, 2, \ldots, n \), with the convention \( \omega_0 = 1 \).

The result (5.7) is interesting from an algorithmic viewpoint for the following reason. To determine the sequence of symmetric predictor polynomials \( s_k(z) \) by means of the recurrence relation (4.1) it suffices to perform a unique inner product per iteration stage \((k - 1 \rightarrow k)\). In fact, the proposed method computes \( \omega_k \) from \( s_{k-1}(z) \) and uses the value \( \omega_{k-1} \) obtained at the preceding stage to determine the coefficient \( \beta_k \) involved in relation (4.1) with the help of formula (5.7). This computation scheme is analogous to that of the split Levinson algorithm \(^8\).

The ordinary predictor polynomial \( a_n(z) \) can be obtained from the last two symmetric predictors, \( s_n(z) \) and \( s_{n-1}(z) \), by means of the appropriate relation (3.3), provided that the corresponding squared norm \( \sigma_n \) is available. To determine this number from the coefficients \( \beta_k \) (and the 'initial value' \( \omega_0 \)), we can make use of formulas (4.2), which yield the recurrence relation

\[ \phi_k = \phi_{k-1}^{-1} + (-1)^{k-1} \beta_k, \tag{5.8} \]

with \( \phi_{2j} = \sigma_{2j} \) and \( \phi_{2j+1} = \sigma_{2j+1}^{-1} \) as in Sec. 4.

Let us now summarize the main part of the resulting linear prediction algorithm. The internal variables are the real numbers \( s_{k,i} \) and \( \phi_k \). The \( k \)th stage consists of the following operations. First, compute the parameter \( \omega_k \) by the inner product relation (5.5), and determine the recurrence coefficient \( \beta_k \) by formula (5.7). Then, compute \( s_k(z) \) and \( \phi_k \) (from \( s_{k-1}(z) \), \( s_{k-2}(z) \) and \( \phi_{k-1} \)) by use of the recurrence relations (4.1) and (5.8). The initial values are

\[ s_{-1}(z) = 0, \quad s_0(z) = 1 + z, \quad \omega_0 = 1, \quad \phi_0 = c_0(0). \tag{5.9} \]

The desired predictor polynomial \( a_n(z) \) is obtained, in the final stage, through the relevant formula (3.3), where the squared norm \( \sigma_n \) is equal to \( \phi_n \) if \( n \) is even and to \( \phi_n^{-1} \) if \( n \) is odd.
In fact, to complete the description of the algorithm we have to specify how to determine the entries $c_k(i)$ of the current Toeplitz matrix $C_k$ which are involved in the inner product formula (5.5). Assume that the numbers $c_{k-1}(i)$ are available (from the preceding stage), and that the diagonal entry $c_k(0)$ is known. Then the desired entries $c_k(i)$ can be computed by solving eq. (2.2), which yields the simple recurrence

$$c_k(i + 1) = 2c_k(i) - c_k(i - 1) - c_{k-1}(i),$$

(5.10)
with $c_k(0)$ and $c_k(1) = c_k(0) - c_{k-1}(0)/2$ as initial values. As to the numbers $c_k(0)$ themselves, they can be obtained from the data $c_n(i)$, in a preliminary stage, by direct use of relation (2.2), with $\zeta_k = 1$.

In the antisymmetric case ($\zeta_k = -1$), we can derive a very similar algorithm based on the recurrence (4.12) for the antisymmetric predictor polynomials $t_k(z)$, with the initialization $t_{-1}(z) = 0$ and $t_0(z) = 1 - z$. Roughly speaking, this method is obtained from the preceding method by formally replacing $z$ by $-z$ in all equations. The recurrence coefficient $\beta_k$ can be computed by the same formula (5.7) as above, where $\omega_k$ is defined as the inner product

$$\omega_k = \sum_{i=0}^{k} c_k(i)t_{k-1,i},$$

(5.11)
for $k = 1, \ldots, n$, and $\omega_0 = 1$. The proof of this key result is basically the same as in the symmetric case, and details are omitted. (Note that the 'antisymmetric version' of the expressions (5.6) results from replacing $a_k(1)$ by $a_k(-1)$ in their right-hand sides. Equivalently, we have $\omega_k = 2/t_k(-1)$.)

The squared norms $\sigma_k$ can be determined from the coefficients $\beta_k$ by means of the recurrence (5.8), with the appropriate initial value $\phi_0 = c_0(0)$; the definition of the numbers $\phi_k$ is formally the same as above, i.e. $\phi_k = \sigma_k$ for even $k$ and $\phi_k = \sigma_k^{-1}$ for odd $k$.

At the end of the procedure, the classical predictor polynomial $a_n(z)$ associated with the given Toeplitz matrix $C_n$ can be computed from the antisymmetric predictors $t_n(z)$ and $t_{n-1}(z)$ by means of the appropriate formula (3.8).

The entries $c_k(i)$ of the relevant Toeplitz matrix $C_k$, involved in the inner product formula (5.11), are computable by use of the recurrence relation

$$c_k(i + 1) = -2c_k(i) - c_k(i - 1) + c_{k-1}(i),$$

(5.12)
for $i = 1, \ldots, k - 1$, together with $c_k(1) = -c_k(0) + c_{k-1}(0)/2$; the initial values
Ck(0) are obtainable from the data cn(i) with the help of relation (2.2), where ζk = -1.

Remark
The algorithms described above (which are 'split versions' of the generalized Levinson algorithm \(^1\)) with ζk = ±1) can be interpreted in the context of the split Levinson algorithm \(^8\,\,17\,\,18\). This relationship stems from the following observation. (In the following we assume ζk = 1; we could make quite similar comments in the other case, ζk = -1.) By definition, the polynomials \((1 - z)s_{2j-1}(z)\) and \(s_{2j}(z)\) are proportional to the antisymmetric and symmetric 'singular predictor polynomials', of degrees 2j and 2j + 1, associated with the Toeplitz matrix \(C_{2j}\) in the framework of the split Levinson algorithm \(^8\). (Note that these polynomials vanish at the points z = 1 and z = -1 respectively.) Analogously, the polynomials \(s_{2j+1}(z)\) and \((1 - z)s_{2j}(z)\) are normalized versions of the symmetric and antisymmetric singular predictor polynomials, of degrees 2j + 1 and 2j + 2, associated with \(C_{2j+1}\). (They vanish for z = -1 and z = 1 respectively.) For this reason, the linear prediction algorithm based on the recurrence relation (4.1) can be interpreted as a 'deflated split Levinson algorithm' in the sense alluded to in ref. 17. Note that it is more complex than the split Levinson algorithm itself. We shall not go into further details about this subject.

REFERENCES
Three-term recurrence in Toeplitz system theory

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