AN OPTIMIZATION PROBLEM IN REFLECTOR DESIGN

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Abstract

In this paper we present the mathematical solution to a problem in the design of cylindrically symmetric reflectors which, when combined with a linear light source, produce a prescribed luminous intensity distribution. Usually there are many such reflectors and one may try to meet design constraints on the dimensions of the reflector, so we consider the following problem. What are the minimum and the maximum value of the ratio of the distances to the light source of the two edges of the reflector surface, under the condition that the reflector realizes the prescribed distribution?

It is shown that this problem admits the following mathematical formulation: what are the extreme values of the functional \( J(\theta) = \int_{t_1}^{t_2} f(s + \theta(s)) \, ds \) over all \( \theta: [t_1, t_2] \to \mathbb{R} \) having a prescribed smooth non-decreasing rearrangement \( \theta' \)? Here \( f \) is a given smooth, odd function with convex, non-negative derivative (in the reflector design problem we have \( f(t) = \tan(t/2) \)). This problem is shown to have a solution of bounded variation when \( \| \theta' \|_{\infty} < 2 \), but may fail to have such a solution when \( \| \theta' \|_{\infty} \geq 2 \). The optimizers \( \theta \) can be described analytically under the conditions that they are of bounded variation and that \( \theta'(t) = \frac{1}{2} \) for only finitely many \( t \). For instance, under these conditions, it is shown that the maximizing \( \theta \) is V-shaped and continuous on the left leg of the V, continuous with the exception of at most finitely many points on the right leg of the V. We work out some examples with relevance to the reflector design problem.

Keywords: constrained optimization, inverse problems, matching, rearrangement, reflector design.

1. Introduction

1.1. Motivation

In this paper a mathematical problem with applications to reflector design is solved. Reflector design problems occur in many lighting and heating
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Fig. 1. A linear light source with a cylindrical reflector.

applications, such as road or playground lighting, car lighting, liquid crystal display backlighting, projection television, oven design, etc. The problem in its most general form is the problem of designing a reflector for a fixed light source such that a prescribed intensity distribution is realized, while, at the same time, certain design specifications on the dimensions of the optical system are met.

In this generality, the problem is far too difficult to admit an analytic solution; such a solution can only be obtained when certain simplifying assumptions are made. The assumptions we make here are that we have linear light sources with cylindrically symmetric reflectors, and that the screen to be illuminated is at an infinite distance. Since under these conditions there are usually many reflectors realizing the required distribution, we are led to investigate their possible dimensions. In this paper we shall derive bounds on the dimensions of the reflectors realizing a given distribution. Although our assumptions are restrictive for most applications, the solution of the problem for a linear light source can provide considerable insight into the solution of the general problem. Indeed, the results of this paper have already proved to provide useful "rules of thumb" for proximate lighting tasks, such as liquid crystal display backlighting, as well.

We start with a more mathematical description of the above-mentioned model problem. We consider linear light sources and cylindrically symmetric reflector surfaces as depicted in fig. 1, so that the light source coincides with the z-axis, and the reflector surface is described by its radial distance function

\[ r = r(t), \quad r = (x^2 + y^2)^{1/2}, \quad t_1 \leq t \leq t_2 \]  

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see fig. 2. The incident ray \((r \cos t, r \sin t)\) is reflected in accordance with the law of reflection, so that

\[
\frac{\dot{r}(t)}{r(t)} = \tan \left[ \frac{t + \theta(t)}{2} \right], \quad t_1 \leq t \leq t_2,
\]

if \(\dot{r}(t)\) exists, where \(\theta(t)\) is the angle corresponding to the reflected beam; see fig. 3. See also ref. 1, where (2) is shown to be a solution to the cylindrical reflector design problem, once an increasing reflected angle function \(\theta(t)\) is known. Indeed, given such a function \(\theta(t)\), \(t_1 \leq t \leq t_2\), the solution

\[
r(t) = r(t_1) \exp \left\{ \int_{t_1}^t \tan \left[ \frac{s + \theta(s)}{2} \right] \, ds \right\}, \quad t_1 \leq t \leq t_2
\]

to the differential equation (2) describes a reflector with \(\theta(t)\) as reflected angle function.

In many practical design problems one is interested in the value distribution function associated with \(\theta(t)\), rather than the actual mapping \(\theta(t)\). That is to say, one is only interested in the amount of light leaving the reflector under various angles, and not in the precise points \((t, r(t))\) on the surface where this light comes from. Hence we are interested for all angles \(\phi_1, \phi_2\) with \(\phi_1 < \phi_2\) in the size of the set of all \(t\) between \(t_1\) and \(t_2\) such that \(\theta(t)\) lies between \(\phi_1\) and \(\phi_2\).

Fig. 3. An illustration of the law of reflection.
A mathematically interesting, and technically relevant, problem is to find out which values $r(t_2)/r(t_1)$ can assume, provided that $\theta$ has a given value distribution function. In this paper we solve this problem under the further assumptions that both the reflection coefficient of the reflector and the luminous intensity of the light source do not vary with $t$; these are the same assumptions under which part of the problem, to be detailed below, was investigated by the authors in ref. 2.

We will now give an example of a result that can be obtained from the results of this paper. To this end, consider a required luminous intensity distribution

Fig. 4. The reflectors realizing a uniform intensity distribution on the interval $[-\pi/6, \pi/6]$ by $a)$ divergent and $b)$ convergent ray bundles.
for the reflected light, described by the function $I(\theta) = 4$ on the interval $[-\pi/6, \pi/6]$. Suppose it to be realized by a reflector that is located between angles $[-2\pi/3, 2\pi/3]$, which has a lower endpoint at a fixed distance $r(-2\pi/3) = 1$. (Here the $t$ and $\theta$ angles are measured as indicated in fig. 3, see also the figures below.) From the method described in ref. 2 one easily finds two "canonical" solutions to this problem, those with convergent and divergent ray bundles, respectively. The functions $\theta(t) = t/4$ and $\theta(t) = -t/4$ describe these bundles. The solutions are shown in fig. 4. Since the two solutions are symmetric, they both have endpoints at distance 1 from the source. There are, however, many other solutions, and it can be shown that $r(2\pi/3)$ is maximum in the case that

$$\theta(t) = \theta_{\text{max}}(t) = \begin{cases} -t/3 - \pi/9 & \text{if } t \in [-2\pi/3, \pi/6], \\ t - \pi/3 & \text{if } t \in [\pi/6, 4\pi/9], \\ t/4 & \text{if } t \in [4\pi/9, 2\pi/3]. \end{cases}$$ (4)

In fig. 5, a graph of this function is drawn. Note that this function is V-shaped, i.e. the left part of the graph is decreasing, and the right part of the graph is increasing. For the ray bundle this means that the lower part is convergent, and the upper part is divergent. This notion of V-shapedness will turn out to be a very important one. Also, $r(2\pi/3)$ is minimum in the case that $\theta(t) = -\theta_{\text{max}}(t)$. The two solutions are shown in fig. 6. The reflectors of all four solutions described above are illustrated in fig. 7.
Fig. 5. Graph of the function $\theta(i)$ of eq. (4), describing the relation between incident and reflected rays, which leads to a reflector with maximum endpoint distance $r(2\pi/3)$.

Fig. 6. The reflectors and ray bundles with (a) maximum and (b) minimum endpoint distances.
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Fig. 6. Continued.

To prove the statements in the above example, considerable mathematical effort is needed. It is this mathematical effort that is described here, rather than the practical implementation of the methods. The latter will be discussed at a more appropriate place. We hope that the above example has given a flavour of the practical applicability of the results of this paper, of which the remainder is written in purely mathematical terms.

1.2. Mathematical Problem Formulation and Summary of Results

In strict mathematical terms we will consider the maximization and the minimization of the functional

\[ J(\theta) := \int_{t_1}^{t_2} f[s + \theta(s)] \, ds \]  

over all measurable functions \( \theta: [t_1, t_2] \to \mathbb{R} \) that are equimeasurable with a prescribed smooth function \( \overline{\theta}: [t_1, t_2] \to \mathbb{R} \). Here \( f \) is a smooth, odd function.
Fig. 7. Four reflector surfaces that realize the same intensity distribution.
Fig. 8. An example of the reflectors that correspond to (a) the non-decreasing and (b) the non-increasing rearrangements of $\vartheta$, respectively.
with convex, non-negative derivative $f'$, such as $f(t) = \tan (t/2)$ on $(-\pi, \pi)$. The condition of equimeasurability means that for all $\phi_1, \phi_2$ with $\phi_1 < \phi_2$ the sets of all $t$ with $\phi_1 < \theta(t) < \phi_2$ and with $\phi_1 < \theta(t) < \phi_2$ have equal Lebesgue measure. For definiteness we always take $\theta$ to be non-decreasing and thus ask for the extreme values of $J(\theta)$ over all $\theta$ having $\theta$ as their common non-decreasing rearrangement. We refer to ref. 3, Secs 10.12–16 for more details concerning rearrangements.

We shall concentrate in this paper on the maximization of (5); the minimization of (5) is easily transformed into a maximization problem of the considered type by replacing $t_2$ by $-t_1$, $t_1$ by $-t_2$, and $\theta(t)$ by $-\theta(-t)$ for $-t_2 \leq t \leq -t_1$. The answer to the maximization problem is particularly easy in two special cases, viz. when $t_1 + \theta(t_1) \geq 0$ or $t_2 + \theta(t_2) \leq 0$. We will show that when $t_1 + \theta(t_1) \geq 0$, then we have

$$\int_{t_1}^{t_2} f[s + \theta(s)] ds \leq \int_{t_1}^{t_2} f[s + \theta(s)] ds,$$

and that when $t_2 + \theta(t_2) \leq 0$, we have

$$\int_{t_1}^{t_2} f[s + \theta(s)] ds \leq \int_{t_1}^{t_2} f[s + \theta(t_1 + t_2 - s)] ds$$

for all allowed $\theta$. Hence the non-decreasing rearrangement $\theta(s)$ and the non-increasing rearrangement $\theta(t_1 + t_2 - s)$ solve the respective maximization problems. In fig. 8 we show an example of the resulting reflector surfaces described by (1) and (2) for these extreme cases. This figure has been borrowed from ref. 2, where an elegant proof for the special case that $f(t) = \tan (t/2)$ is presented. In Sec. 3.1 of the present paper we derive a similar result for more general functions $f$.

Unfortunately, the results for the case that $t_1 + \theta(t_1) < 0 < t_2 + \theta(t_2)$ are not so easy to state, and the proofs are in keeping with it. Firstly, it may very well happen that the maximization problem does not admit a solution in the space of all measurable functions $\theta$ having $\theta$ as their common non-decreasing rearrangement. Also, in the cases that there does exist a solution, it may be discontinuous at many places and its actual form, which can be rather complicated, usually depends on $f$. (If $\theta$ has $n$ discontinuities, then the corresponding reflector will consist of $n + 1$ smooth facets.)

On the other hand we have certain not too restrictive conditions under which we can show the optimal $\theta$s to be reasonably well behaved (what this means will be explained below). These conditions are that $\theta'(t) < 2$ for all
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t \in [t_1, t_2], and that \( \partial'(t) = 1/2 \) for only finitely many points \( t \in [t_1, t_2] \). In reflector design applications, both these conditions are usually satisfied.

Let us summarize the results of this paper. In Sec. 2 we consider the discrete version of the maximization problem. That is, given increasing sequences \( s_1, \ldots, s_n \) and \( \partial_1, \ldots, \partial_n \), together with a smooth, odd function \( f \) for which \( f' \) is (strictly) convex and non-negative, then we want to maximize

\[
J(\theta) := \sum_{i=1}^{n} f[s_i + \theta(s_i)], \quad (8)
\]

over all bijections \( \theta: \{s_1, \ldots, s_n\} \to \{\partial_1, \ldots, \partial_n\} \). This problem can be seen as a matching problem, and it is solvable in \( O(n^3) \) time. However, because of the conditions on \( f \), several properties of optimizers can be deduced. For instance, denoting \( f_i(s) = f_i(s_i) \), it will be shown that for any maximizer \( \theta \) and any \( k, m \) with \( 1 \leq k \leq m \leq n \) we have

\[
\{\theta_k, \theta_m\} \cap \{\min_{1 \leq i \leq m} \theta_i, \max_{1 \leq i \leq m} \theta_i\} \neq \emptyset. \quad (9)
\]

Furthermore, when

\[
\max_{1 \leq i \leq n-2} (\partial_{i+1} - \partial_i) < \min_{1 \leq i \leq n-2} (s_{i+1} - s_i), \quad (10)
\]

it turns out that for any maximizer and any \( k, m \) with \( 1 \leq k \leq m \leq n \) we have

\[
\{\theta_k, \theta_m\} \cap \{\max_{1 \leq i \leq m} \theta_i\} \neq \emptyset \quad (11)
\]

The latter property is equivalent with \( \theta \) being V-shaped: there is an \( n_0 \) with \( 1 \leq n_0 \leq n \), such that

\[
\theta_1 > \theta_2 > \cdots > \theta_{n_0-1} > \theta_{n_0} = \partial_1 < \theta_{n_0+1} < \cdots < \theta_{n-1} < \theta_n. \quad (12)
\]

(If \( \theta \) has the converse property, i.e. if \( -\theta \) is V-shaped, then \( \theta \) is usually said to be unimodal.) Also, the deviation of the maximizing \( \theta \) from being V-shaped in the general case can be quantified in terms of the extent to which (10) is violated; see Proposition 2.7. Finally we show that, under the condition that

\[
\max_{1 \leq i \leq n-2} (\partial_{i+1} - \partial_i) < \min_{1 \leq i \leq n-1} (s_{i+1} - s_i), \quad (13)
\]

we can even solve the discrete problem by a greedy \( O(n) \) algorithm.

In Sec. 3.1 we present existence results for the maximization of

\[
J_\phi(\theta) := \int_{t_1}^{t_2} f[\phi(s) + \theta(s)] \, ds, \quad (14)
\]

over all measurable \( \theta \) having \( \partial \) as their common non-decreasing rearrange-
ment. Here $\phi$ is a given bounded function. We shall show the result announced in connection with (6) and (7) for the case that $\phi(t), \theta(t) \geq 0$ for all $t \in [t_1, t_2]$. Furthermore we show the following. Let the variation $\mathrm{Var}(\theta; t_1, t_2)$ of $\theta$ over $[t_1, t_2]$ be defined by

$$\mathrm{Var}(\theta; t_1, t_2) = \sup \left\{ \sum_{k=1}^{n-1} |\theta(s_{k+1}) - \theta(s_k)| \mid \text{such that} \right. t_1 = s_1 < s_2 < \cdots s_n = t_2; \ n \in \mathbb{N} \left. \right\}. \quad (15)$$

Then we show the following: for any $V \geq \theta(t_2) - \theta(t_1)$ there exists an allowed $\theta_v$ with $\mathrm{Var}(\theta_v; t_1, t_2) \leq V$ such that $J_\theta(\theta_v) \geq J_\theta(\theta)$ for all allowed $\theta$ with $\mathrm{Var}(\theta; t_1, t_2) \leq V$. Although in actual reflector design problems the restriction to mappings $\theta$ of finite variation is quite natural, this existence result is unsatisfactory in the sense that it does not exclude (and indeed, it happens) that $\mathrm{Var}(\theta_v; t_1, t_2) \to \infty$ as $V \to \infty$. A further result that we present in Sec. 3.1 is that $\sup_\theta J_\theta(\theta) = \sup_\phi J_\theta(\phi)$, where the suprema are over all $\theta$ and $\phi$ with common non-decreasing rearrangements $\theta$ and $\phi$, respectively. This result is useful when one of the optimizations is easier than the other. Finally, a result is presented showing that the continuous problem can be considered as a limit case of the discrete problem, so that the results of Sec. 2 can be carried over to the continuous problem.

In Sec. 3.2 we consider the case $\phi(s) = s$ for all $s$ in (14) in more detail, and we analyze the optimizers $\theta$ under the condition that their variation (15) is finite. For instance, it is shown that these optimizers are V-shaped. Also, with the aid of Sec. 3.1 it is shown that there exist optimizers of finite variation whenever $\bar{\theta}'(s) < 2$ for all $s \in [t_1, t_2]$. Furthermore, it is shown that V-shaped optimizers are continuous on the left leg of the V and that the number of discontinuities of $\theta$ on the right leg is bounded from above in terms of the number of $s$ with $\bar{\theta}'(s) = \frac{1}{2}$, if that number is finite. For instance, when $\bar{\theta}'(s) < \frac{1}{2}$ for all $s \in [t_1, t_2]$, we find that the optimizer $\theta$ is continuous. Also, the form of $\theta$, both on the left leg and between the discontinuities on the right leg, is determined analytically in terms of the discontinuities of $\theta$ and the values of $\theta$ assumed at $t_1$ and $t_2$. This allows us to express $J(\theta)$ as a finite series of integrals involving known functions, with integration bounds that are to be chosen so as to yield the highest possible value for $J(\theta)$. The latter problem can get quite complicated.

In Sec. 4 we present, again under the condition that the optimizers are of finite variation, analytical results for the case that there is at most one point $s$ with $\bar{\theta}'(s) = \frac{1}{2}$. (In reflector design problems, this will often be the case.)
Finally, in Sec. 5 we present examples, both for the discrete and the continuous case, some of which are relevant to the reflector design problem. These examples also serve to illustrate a curious duality between the existence of non-injective solutions of the continuous problem when \( \vartheta'(s) < \frac{1}{2} \) for all \( s \), and the non-existence of solutions of this problem when \( \vartheta'(s) \geq 2 \) is allowed to occur.

2. The discrete problem

2.1. The discrete problem seen as a matching problem

In this section we consider an increasing sequence \( s_1, \ldots, s_n \) and an increasing sequence \( \vartheta_1, \ldots, \vartheta_n \), together with a smooth, odd function \( f \) for which \( f' \) is (strictly) convex and non-negative, and we want to maximize

\[
J(\theta) := \sum_{i=1}^{n} f[s_i + \theta(s_i)],
\]  

over all bijections \( \theta: \{s_1, \ldots, s_n\} \rightarrow \{\vartheta_1, \ldots, \vartheta_n\} \). This problem is a special case of a well-known matching problem. In order to formulate this matching problem, we briefly recall some notions from graph theory. For more details, we refer to ref. 4.

A graph \( G = (V, E) \) is called bipartite if \( V = A \cup B \) for two disjoint non-empty subsets \( A \) and \( B \) of \( V \) such that all edges in \( E \) join a vertex of \( A \) to a vertex of \( B \). A bipartite graph is called complete if each vertex in \( A \) is adjacent to each vertex in \( B \). A subset of edges \( M \subset E \) of a graph \( G \) is called a matching of \( G \) if no two edges in \( M \) have a vertex in common. A matching \( M \) is called perfect if each vertex is covered by an edge in \( M \). If the graph \( G \) is weighted, i.e. if each edge \( e \in E \) has a weight \( w_e \in \mathbb{R} \) associated with it, then the weight of a matching \( M \) is defined to be \( \Sigma_{e \in M} w_e \). A maximum weight perfect matching is a perfect matching that has the greatest weight among all perfect matchings.

It is easily seen that maximizing (16) is precisely the problem of finding a maximum weight perfect matching in a weighted complete bipartite graph. Specifically, let \( A = \{s_1, s_2, \ldots, s_n\}, B = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\}, E = \{(s_i, \vartheta_j) | s_i \in A, \vartheta_j \in B\}, \) and \( w_e = f(s_i + \vartheta_j) \) for \( e = (s_i, \vartheta_j) \). This problem is solvable in polynomial time: Gabow\(^5\), Lawler\(^6\), and Cunningham and Marsh\(^7\) have developed algorithms which take \( O(|V|^3) \) time. In our problem however, the "weight function" has some special properties. From this we can deduce several properties of optimal mappings \( \theta \) (i.e. optimal matchings). It might be interesting to investigate whether these properties may lead to a faster match-
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ing algorithm for these special weight functions. This topic, however, is not addressed in this paper.

What is more important for this paper is that the results of this section provide us with insight as to when the continuous problem is solvable (and when not), and what the optimal $\theta(t)$ looks like. It is also for this reason that the above problem is formulated as that of finding an optimal mapping, rather than one of finding an optimal permutation, another way to present the problem which would have emphasized the symmetry of the problem (in the sense that the $s_i$s and the $\bar{\theta}_i$s play similar roles). Finally, the restriction to increasing sequences is for convenience only; the results of this section can be applied to non-decreasing sequences as well.

2.2. Basic properties of maximizers

From now on, we assume that $\theta$ is a bijection that maximizes (16), and we will write $\theta_i = \theta(s_i)$ for all $i \in \{1, \ldots, n\}$. In this subsection we will see that $\theta$ maps at least one of the extreme points $S_1, S_2$ onto one of the extreme points $\bar{\theta}_1, \bar{\theta}_n$. We will also investigate conditions under which any of these situations may occur. The following result is basic to the remainder of this section.

Proposition 2.1

Let $1 \leq k < l \leq n$. Then we have
(a) $\theta_k < \theta_l \Rightarrow - (\theta_k + \theta_l) \leq s_k + s_l$,
(b) $\theta_k > \theta_l \Rightarrow - (\theta_k + \theta_l) \geq s_k + s_l$.

Proof

By optimality of $\theta$, we have

$$f(s_k + \theta_k) + f(s_l + \theta_l) \geq f(s_k + \theta_l) + f(s_l + \theta_k).$$

The assumptions on $f$ (see the beginning of Sec. 2.1) imply that the function

$$\phi_{n,t}(s) = f(s + \eta) - f(s + \tau), \quad s \in \mathbb{R},$$

is even, i.e. it is symmetric about the point $s = -(\eta + \tau)/2$. Furthermore, $\phi_{n,t}$ is positive and strictly convex when $\eta > \tau$, negative and strictly concave when $\eta < \tau$. Hence (17) together with $s_k < s_l$ imply that

$$-[s_i + (\theta_k + \theta_l)/2] \leq s_k + (\theta_k + \theta_l)/2,$$

or

$$-[s_i + (\theta_k + \theta_l)/2] \geq s_k + (\theta_k + \theta_l)/2,$$

according to whether $\theta_k < \theta_l$ or $\theta_k > \theta_l$, as required. 

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The next result involves three different points; we present only the most significant conclusions that can be drawn concerning three points.

**Proposition 2.2**
Let \( 1 \leq k < l < m \leq n \). Then we have

(a) \( \theta_k > \theta_m > \theta_l \Rightarrow \theta_k - \theta_l \leq s_l - s_k \),

(b) \( \theta_l > \theta_k > \theta_m \Rightarrow \theta_k - \theta_m \geq s_m - s_k \),

(c) \( \theta_l > \theta_m > \theta_k \) does not occur.

**Proof**
(a) Because \( \theta_k > \theta_m \) and \( \theta_m > \theta_l \) it follows from Proposition 2.1 that

\[
-(\theta_k + \theta_m) > s_k + s_m \quad \text{and} \quad -(\theta_l + \theta_m) < s_l + s_m. \tag{21}
\]

By combining these two inequalities, implication (a) follows.

(b) Because \( \theta_l > \theta_k \) and \( \theta_l > \theta_m \) it follows from Proposition 2.1 that

\[
-(\theta_k + \theta_l) < s_k + s_l \quad \text{and} \quad -(\theta_l + \theta_m) > s_l + s_m. \tag{22}
\]

By combining these two inequalities, implication (b) follows.

(c) In the proof of (b) the fact that \( \theta_k > \theta_m \) was not used, but this follows already from the right hand side of the implication (b). Hence \( \theta_l > \theta_m > \theta_k \) does not occur. \( \square \)

The next result is an important characteristic of a maximizing \( \theta \). It says that \( \theta \) maps at least one of the extreme points \( s_1, s_n \) onto one of the extreme points \( \overline{s}_i, \overline{s}_n \).

**Theorem 2.3**
For a maximizing \( \theta \) we have

\[
\{\theta_1, \theta_n\} \cap \{\overline{s}_i, \overline{s}_n\} \neq \emptyset. \tag{23}
\]

**Proof**
Suppose that (23) is not true. Then there are \( i, j \) with \( 1 < i, j < n \) such that

\[
\theta_i = \overline{s}_i < \theta_1 \quad \text{and} \quad \theta_j = \overline{s}_n > \theta_n. \tag{24}
\]

It follows from Proposition 2.2(b) and 2.2(c) with \( k = 1, l = j, m = n \) that

\[
\theta_1 - \theta_n \geq s_n - s_1. \tag{25}
\]
Next it follows from Proposition 2.2(a) with \( k = 1, l = i, m = n \) that

\[
\theta_1 - \bar{\theta}_1 \leq s_i - s_1. \tag{26}
\]

However, \( s_i < s_n \) and \( \theta_n > \bar{\theta}_1 \), and this shows that (25) and (26) yield a contradiction.

An immediate consequence of this proposition is the following.

**Corollary 2.4**

Let \( 1 \leq k < m \leq n \). Then we have

\[
\{ \theta_k, \theta_m \} \cap \{ \min_{k \leq i \leq m} \theta_i, \max_{k \leq i \leq m} \theta_i \} \neq \emptyset. \tag{27}
\]

Now that we know that an optimal \( \theta \) "matches" at least one pair of extremal points, we can investigate necessary and sufficient conditions under which any of these matchings occur. The following proposition gives some of these conditions.

**Proposition 2.5**

For a maximizing \( \theta \) we have

(a) \(- (\theta_1 + \theta_2) < s_1 + s_2 \Rightarrow \theta_1 = \bar{\theta}_1,\)

(b) \(- (\theta_{n-1} + \theta_n) > s_1 + s_n \Rightarrow \theta_1 = \bar{\theta}_n,\)

(c) \(- (\theta_1 + \theta_n) > s_{n-1} + s_n \Rightarrow \theta_n = \bar{\theta}_1,\)

(d) \(- (\theta_1 + \theta_n) < s_1 + s_n \Rightarrow \theta_n = \bar{\theta}_n,\)

(e) \(- (\theta_{n-1} + \theta_n) < s_1 + s_2 \Rightarrow \theta_1 \neq \bar{\theta}_n,\)

(f) \(- (\theta_1 + \theta_2) > s_1 + s_n \Rightarrow \theta_1 \neq \bar{\theta}_1.\)

**Proof**

We will only prove (a) and (b). The rest is proven similarly.

(a) Suppose that the left member of the implication (a) is valid, and that \( \theta_1 > \bar{\theta}_1 = \theta_k \) for some \( k > 1 \). Then by Proposition 2.1(b) we have

\[
-(\bar{\theta}_1 + \bar{\theta}_2) \geq -(\theta_1 + \theta_k) \geq s_i + s_k \geq s_1 + s_2, \tag{28}
\]

a contradiction.

(b) Suppose that the left member of the implication (b) is valid, and that \( \theta_1 < \bar{\theta}_n = \theta_k \) for some \( k > 1 \). Then by Proposition 2.1(a) we have

\[
-(\bar{\theta}_{n-1} + \bar{\theta}_n) \leq -(\theta_1 + \theta_k) \leq s_1 + s_k \leq s_1 + s_n, \tag{29}
\]

a contradiction.

We conclude this subsection with the discrete analogue of the result in ref. 2, which is generalized in Proposition 3.1.
Corollary 2.6
If \( s_1 + s_2 + 2\theta_1 \geq 0 \) then \( \theta_i = \theta_1 \) for all \( i \in \{1, \ldots, n\} \). If \( s_n + s_{n-1} + 2\theta_n \leq 0 \) then \( \theta_i = \theta_{n+1-i} \) for all \( i \in \{1, \ldots, n\} \).

**Proof**
Repeatedly apply Propositions 2.5(a) and 2.5(c) for the first and second statement, respectively. \( \square \)

2.3. V-shaped maximizers

In the previous subsection we have seen that \( \theta \) satisfies condition (27). From this one can deduce that the number of mappings \( \theta: \{s_1, \ldots, s_n\} \to \{\theta_1, \ldots, \theta_n\} \) that can possibly be a maximizer, is reduced from \( n! \) to \( \left\lfloor \frac{1}{2} (2 + \sqrt{2})^{n-1} \right\rfloor \). Unfortunately, this is the best one can do: for each function \( f \) and for each mapping \( \theta \) that satisfies condition (27), one can find numbers \( s_1, \ldots, s_n \) and \( \theta_1, \ldots, \theta_n \) such that \( \theta \) is a maximizer (the proof uses Proposition 2.5 and induction).

Nevertheless, there is a preference for the \( \theta \)s to be V-shaped. By this we mean that there is an \( n_0 \) with \( 1 \leq n_0 \leq n \), such that

\[
\theta_1 > \theta_2 > \cdots > \theta_{n_0-1} > \theta_{n_0} = \theta_1 < \theta_{n_0+1} < \cdots < \theta_{n-1} < \theta_n. \tag{30}
\]

Note that \( \theta \) is V-shaped if and only if for all \( k, m \) with \( 1 \leq k \leq m \leq n \), we have (compare with (27))

\[
\max_{k \leq l \leq m} \theta_l \in \{\theta_k, \theta_m\}. \tag{31}
\]

We will now show that the deviation from being V-shaped puts strong restrictions on the sets \( \{s_1, \ldots, s_n\} \) and \( \{\theta_1, \ldots, \theta_n\} \). To this end, first note that if \( \theta \) is not V-shaped, then there is an \( l \) with \( 1 < l < n \) such that \( \theta_1 > \theta_{l-1} \) and \( \theta_l > \theta_{l+1} \). From Proposition 2.2 it then follows that \( \theta_i > \theta_{l-1} > \theta_{l+1} \). The following proposition is formulated, mainly for convenient graphical display in fig. 9, for the case \( s_i = i \) for all \( i \). The generalization to the case of arbitrary increasing \( s_i \) is straightforward.

**Proposition 2.7**

Let \( s_i = i \) for all \( i \in \{1, \ldots, n\} \). Assume that \( \theta \) is not V-shaped, so that there is an \( l \) with \( 1 < l < n \) such that \( \theta_1 > \theta_{l-1} > \theta_{l+1} \). Let \( i \) and \( j \) be such that

\[
\theta_i + i = \min_{k \leq l, \theta_k < \theta_l} (\theta_k + k), \quad \text{and} \quad \theta_j = \min_{k \leq l} \theta_k. \tag{32}
\]

Note that, by definition \( i \leq j \). Now put \( \alpha = \theta_i + i - \theta_j \). Then the graph \( \{(k, \theta_k) | k \in \{1, \ldots, n\}\} \) of \( \theta \) is restricted to the grey regions in fig. 9. More
Fig. 9. In the case that θ is not V-shaped, its graph is restricted to the grey regions.

precisely, we have
(a) $k < α \Rightarrow θ_i < θ_k ≤ θ_l + α - k$,
(b) $α < k < l \Rightarrow θ_l > θ_k \geq \max \{θ_j, θ_l + α - k\}$,
(c) $l < k \Rightarrow θ_k > θ_l$ or $θ_k ≤ θ_l + α - k$.

Moreover, we have
(d) $l < k < m; θ_l > θ_k$ and $θ_l > θ_m = θ_k > θ_m$.

Finally, θ is increasing between j and l, and decreasing before α.

Proof
(a) Let $k < α$. Suppose that $θ_k < θ_l$. Then

$$θ_k + k < θ_k + α = θ_k + θ_l + i - θ_l < θ_l + i,$$

conflicting the definition of i. So $θ_k > θ_l$. Now assume $θ_k > θ_l$. Then $θ_k > θ_l > θ_i$, so that $θ_k - θ_l ≤ i - k$ by Proposition 2.2(a), i.e. $θ_k ≤ θ_l + α - k$.

(b) Let $α < k < l$. By definition of i and j, we have $θ_k \geq \max \{θ_j, θ_l + α - k\}$. This leaves us to show that $θ_k < θ_l$. To this end, first suppose that we have a $k$ with $α < k < i$ and $θ_k > θ_l$. Then $θ_k > θ_l > θ_i$, so that by Proposition 2.2(a) we have $θ_k - θ_l ≤ i - k$, i.e. $θ_k ≤ θ_l + $
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\[ \alpha - k \leq \theta_i, \] a contradiction. Next, suppose we have a \( k \) with \( i < k < l \) such that \( \theta_k > \theta_i > \theta_l \). Then by Proposition 2.2(b) we have \( \theta_i - \theta_l \leq l - i \), i.e. \( \theta_l \leq \theta_i + \alpha - l < \theta_i \), a contradiction.

(c) Let \( l < k \). When \( \theta_k < \theta_i \), we have by Proposition 2.2(b) that \( \theta_i - \theta_k \geq k - i \), i.e. \( \theta_k \leq \theta_i + \alpha - k \).

(d) Let \( l < k < m \) with \( \theta_i > \theta_k \) and \( \theta_i > \theta_m \), and suppose that \( \theta_k < \theta_m \). Then we have by Proposition 2.2(a) that \( \theta_i - \theta_k \leq k - l \), i.e. \( \theta_k \geq \theta_i + l - k > \theta_i + \alpha - k \), which contradicts (c). This proves monotonicity of \( \theta \) in the region to the right of \( l \), below \( \theta_i \).

Finally, monotonicity in the region between \( j \) and \( l \) and before \( \alpha \) are proved similarly, by applying Proposition 2.2.

In the situation of Proposition 2.7 it follows that either \( \theta \) is increasing from \( j \) onwards, or that the sequence \( \theta_n \) exhibits a gap of at least \( l + 1 - j \geq 2 \). Also, when \( n_0 \) is such that \( \theta_{n_0} = \bar{\theta}_1 \), we have that \( \theta \) is increasing from \( n_0 \) onwards (so this holds for any optimal \( \theta \), V-shaped or not). An immediate consequence of the existence of the gap for non-V-shaped maximizers is that \( \theta \) is V-shaped whenever

\[ \max_{1 \leq i \leq n-1} (\bar{\theta}_{i+1} - \bar{\theta}_i) < 2. \] (34)

For arbitrary \( s_i \), the analogue of this is given by the following corollary. Its proof is the same as for the case that \( s_i = i \) for all \( i \).

**Corollary 2.8**
The maximizer \( \theta \) is V-shaped if

\[ \max_{1 \leq i \leq n-1} (\bar{\theta}_{i+1} - \bar{\theta}_i) < \min_{1 \leq i \leq n-2} (s_{i+2} - s_i). \] (35)

In the remainder of this section we shall analyze optimal V-shaped \( \theta \)s somewhat further for the case that \( s_i = i \) for all \( i \). This special case arises naturally when the continuous problem of Sec. 3 is discretized. So assume that \( \theta \) is V-shaped and that \( n_0 \) is such that \( \theta_{n_0} = \bar{\theta}_1 \). The following proposition tells us something about how the points on the left and right leg of the V are relatively situated.

**Proposition 2.9**
Let \( k < n_0 \). There is at most one \( l > n_0 \) such that

\[ \theta_k > \theta_l > \theta_{k+1}. \] (36)

When such an \( l \) exists, then we have for all \( m \geq k \)

\[ \theta_m - \theta_k \geq -(m' - k), \] (37)
in particular, \( \theta_k - \theta_{k+1} \leq 1 \). Furthermore, when \( p \) is such that
\[
\theta_k > \theta_i > \theta_{k+1} > \cdots > \theta_{k+p} > \theta_{i-1} > \theta_{k+p+1},
\]
then we have
\[
\frac{1}{2}p - 1 < \frac{1}{2}(\theta_k + \theta_{k+1}) - \frac{1}{2}(\theta_{k+p} + \theta_{k+p+1}) < \frac{1}{2}p.
\]

**Proof**

Let \( l > n_0 \) be such that (36) holds. It follows from Proposition 2.1 that
\[
-(\theta_k + k) \geq \theta_i + l \geq -(\theta_{k+1} + k + 1) > -(\theta_k + k) - 1.
\]
Now since
\[
\theta_i - 1 + (l - 1) + 1 \leq \theta_i + l \leq \theta_{i+1} + (l + 1) - 1,
\]
we see from (40) that
\[
\theta_{i-1} + (l - 1) \leq -(\theta_k + k) - 1, \quad \text{and} \quad \theta_{i+1} + (l + 1) > -(\theta_k + k).
\]
Hence (40), and therefore (36), is not satisfied when \( l \) is replaced by \( l + 1 \).

The validity of (37) follows from Proposition 2.2(a).

To show (39) from (38), we observe that Proposition 2.1, together with
\[
\theta_k > \theta_i > \theta_{k+1}, \quad \text{and} \quad \theta_{k+p} > \theta_{i-1} > \theta_{k+p+1}
\]
imply that
\[
k + l < -(\theta_k + \theta_{k+1}) < k + l + 1,
\]
and
\[
k + p + l - 1 < -(\theta_{k+p} + \theta_{k+p+1}) < k + p + l.
\]
Combination of these two inequalities yields (39).

We will complete this section with a special case in which a greedy \( O(n) \) algorithm solves the matching problem.

**Proposition 2.10**

Assume that
\[
\max_{1 \leq i \leq n-1} (\bar{\theta}_{i+1} - \bar{\theta}_i) < \min_{1 \leq i \leq n-2} (s_{i+2} - s_i).
\]

Then
\[
(a) \quad -(\bar{\theta}_{n-1} + \bar{\theta}_n) > s_1 + s_n \Rightarrow \theta_1 = \bar{\theta}_n,
\]
\[
(b) \quad -(\bar{\theta}_{n-1} + \bar{\theta}_n) < s_1 + s_n \Rightarrow \theta_n = \bar{\theta}_1.
\]
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Proof
The implication (a) is the same as 2.5(b). To prove (b), assume that 
\(- (\bar{\theta}_{n-1} + \bar{\theta}_n) < s_1 + s_n\) and that \(\theta_n < \bar{\theta}_n\), so \(\theta_n = \bar{\theta}_{n-k}\) for some \(k \geq 1\). Note that (45) implies (35), so \(\theta\) is V-shaped, and consequently, we have \(\theta_k = \bar{\theta}_{n-k+1}\). From Proposition 2.1(b) it then follows that 
\[- (\bar{\theta}_{n-k+1} + \bar{\theta}_{n-k}) \geq s_k + s_n. \tag{46}\]

On the other hand, we have 
\[- (\bar{\theta}_{n-k+1} + \bar{\theta}_{n-k}) = - (\bar{\theta}_n + \bar{\theta}_{n-1}) + (\bar{\theta}_n - \bar{\theta}_{n-2}) + (\bar{\theta}_{n-1} - \bar{\theta}_{n-3}) + \cdots + (\bar{\theta}_{n-k+2} - \bar{\theta}_{n-k}) < (s_n + s_1) + (s_2 - s_1) + (s_3 - s_2) + \cdots + (s_k - s_{k-1}) = s_k + s_n, \tag{47}\]
by (b) and (45). This contradicts (46).

Note that, in the proposition above, in the case that \(- (\bar{\theta}_{n-1} + \bar{\theta}_n) = s_1 + s_n\), we get \(\{\theta_1, \theta_n\} = \{\bar{\theta}_1, \bar{\theta}_n\}\); both possible assignments yield the same value of the functional. It is clear that the above proposition can be applied repeatedly, and we get the following result.

Corollary 2.11
If \[\max_{1 \leq i \leq n-1} (\bar{\theta}_{i+1} - \bar{\theta}_i) < \min_{1 \leq i \leq n-2} (s_{i+2} - s_i), \tag{48}\]
then we can find a maximizer in \(O(n)\) steps.

3. The continuous problem
We consider in this section the maximization of 
\[J(\theta) := \int_{t_1}^{t_2} f(s + \theta(s)) \, ds. \tag{49}\]
Here \(f\) is a smooth, odd function with non-negative, convex derivative \(f'\), and \(\theta\) is equimeasurable with a given, smooth, non-decreasing function \(\bar{\theta}\) defined on \([t_1, t_2]\), i.e. the sets 
\[S_\theta(a, b) := \{t | a < \theta(t) < b\} \quad \text{and} \quad S_{\bar{\theta}}(a, b) := \{t | a < \bar{\theta}(t) < b\} \tag{50}\]
have equal measure for all \(a, b \in \mathbb{R}\). The set of all functions \(\theta\) equimeasurable
with \( \mathcal{E} \) will be denoted \( \mathcal{E} \). In solving this problem we are strongly inspired by the results of Sec. 2 on the discrete problem. However, there are also conspicuous differences between the two problems, the most important one being the non-trivial matter of the existence of maximizers in the continuous problem.

The continuous problem can be discretized so that a discrete problem of the type dealt with in Sec. 2 is obtained. For instance, when \( n \in \mathbb{N} \), then let

\[
\delta^{(n)} := \frac{t_2 - t_1}{2n},
\]

(51)

and let for all \( i \in \{1, \ldots, n\} \)

\[
\delta_i^{(n)} := t_1 + (2i - 1)\delta^{(n)}, \quad \text{and} \quad \mathcal{E}_i^{(n)} := \mathcal{E}(s_i^{(n)}).
\]

(52)

In other words, the interval \([t_1, t_2]\) is divided into \( n \) equally sized subintervals, and the midpoints of these intervals are chosen as \( s_i^{(n)} \). We can now consider the corresponding discrete optimization problem (16). If \( \theta^{(n)} \) is a maximizer of this problem, we can associate with it a step function

\[
\theta_{\text{step}}^{(n)} := \sum_{i=1}^{n} \theta_i^{(n)} I_i^{(n)},
\]

(53)

where \( \theta_i^{(n)} = \theta^{(n)}(s_i^{(n)}) \), and where \( I_i^{(n)} \) is the indicator function of the interval \((s_i - \delta^{(n)}, s_i + \delta^{(n)})\) for all \( i \) with \( 1 < i \leq n \), and \( I_1^{(n)} \) is the indicator function of \([s_1 - \delta^{(n)}, s_1 + \delta^{(n)}]\). Now we can hope that

\[
J(\theta_{\text{step}}^{(n)}) \to \sup \{ J(\theta) \mid \theta \in \mathcal{E} \},
\]

(54)

and that (a subsequence of) \( \theta_{\text{step}}^{(n)} \) converges to a \( \theta \) maximizing \( J(\theta) \) as \( n \to \infty \), when such a \( \theta \) exists.

This section is subdivided as follows. In Subsec. 3.1 we present the solution of the problem in case that \( 0 \notin (t_1 + \mathcal{E}(t_1), t_2 + \mathcal{E}(t_2)) \), which is a generalization of the result in ref. 2. Furthermore, we show existence of a maximizer among all functions \( \theta \in \mathcal{E} \), whose variation

\[
\text{Var} (\theta; t_1, t_2) := \sup \left\{ \sum_{k=1}^{n-1} |\theta(s_{k+1}) - \theta(s_k)| \text{ such that } \right\}
\]

\[
t_1 = s_1 < s_2 < \cdots < s_n = t_2; \quad n \in \mathbb{N}
\]

(55)

does not exceed a prescribed threshold. We will denote by \( \mathcal{E}_V \) the set of all \( \theta \in \mathcal{E} \) with \( \text{Var} (\theta; t_1, t_2) \leq V \). Also, a statement concerning the convergence of the solution of the discretized problem is given and special attention is paid to the case that \( \mathcal{E}'(t) < 2 \), for all \( t \in [t_1, t_2] \).
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In Subsec. 3.2 we assume that there is a $\theta$ with finite variation $\text{Var} (\theta; t_1, t_2)$ such that $J(\theta) \geq J(\eta)$ for all $\eta \in \Theta$. Then we present the basic properties of this $\theta$ such as being V-shaped, and continuity on the left leg of the V. We conclude it with a more detailed analysis of these $\theta$s, and we attempt to describe them and their functional value $J(\theta)$ analytically.

3.1. Some existence results

We present in this subsection some existence results for the slightly more general problem of maximizing (over $\theta$)

$$J_\phi(\theta) := \int_0^1 f[\phi(s) + \theta(s)] \, ds,$$

where $\phi$ has a smooth non-decreasing rearrangement $\bar{\phi}$, see ref. 3, Sec. 10.12. The set of all functions $\phi$ equimeasurable with $\bar{\phi}$ will be denoted $\bar{\phi}$.

Proposition 3.1

Let $f: [0, \infty) \to [0, \infty)$ be a non-decreasing, smooth, convex function with $f(0) = 0$, and let $\phi$ and $\theta$ be two bounded, measurable non-negative functions defined on $[0, 1]$. Then

$$\int_0^1 f[\phi(s) + \theta(s)] \, ds \leq \int_0^1 f[\phi(s) + \theta(s)] \, ds \leq \int_0^1 f[\bar{\phi}(s) + \bar{\theta}(s)] \, ds. \quad (57)$$

Here $\bar{\theta}(s) = \bar{\theta}(1 - s)$ and $\bar{\phi}(s) = \bar{\phi}(1 - s)$ are the non-increasing rearrangements of $\theta$ and $\phi$, respectively.

Proof

For all $t \geq 0$, we have

$$f(t) = \int_0^\infty f''(u) \max(0, t - u) \, du + tf'(0). \quad (58)$$

We can assume that $f'(0) = 0$ since there is equality in (57) for linear $f$s, and then we get by Fubini's theorem

$$\int_0^1 f[\phi(s) + \theta(s)] \, ds = \int_0^\infty f''(u) \left\{ \int_0^1 \max(0, \phi(s) + \theta(s) - u) \, ds \right\} \, du. \quad (59)$$

Since $\max(0, x - u) = \max(u, x) - u$, it suffices to show that

$$\int_0^1 \max(u, \bar{\phi}(s) + \bar{\theta}(s)) \, ds \leq \int_0^1 \max[u, \phi(s) + \theta(s)] \, ds \leq \int_0^1 \max[u, \bar{\phi}(s) + \bar{\theta}(s)] \, ds \quad (60)$$

for any $u \geq 0$. 
We shall first prove (60) for functions $\phi$ and $\theta$ of the form

$$
\phi(s) = \sum_{k=1}^{n} \phi_k I_k(s), \quad \theta(s) = \sum_{k=1}^{n} \theta_k I_k(s),
$$

(61)

where $I_k(s)$ is the indicator function of $[(k-1)/n, k/n]$. In this case the inequality to be proved reduces to

$$
\sum_{k=1}^{n} \max (u, \phi_k + \theta_k) \leq \sum_{k=1}^{n} \max (u, \phi_k + \phi_k) \leq \sum_{k=1}^{n} \max (u, \phi_k + \theta_k)
$$

(62)

for any $u \geq 0$, where $\bar{\phi}_1, \ldots, \bar{\phi}_n$ is the non-decreasing ordering of the sequence $\phi_1, \ldots, \phi_n$, etc.

The elementary inequality

$$
a \leq b \text{ and } c \leq d \Rightarrow \max (u, a + c) + \max (u, b + d) \geq \max (u, b + c) + \max (u, a + d)
$$

(63)

for $u \geq 0$ gives what is required for proving (62). To show (63) we just note that for all $y, u \in \mathbb{R}$, we have

$$
\max (u, y) = \frac{1}{2} (y + u + |y - u|),
$$

(64)

and that the function $x \rightarrow |c + x| - |d + x|$ is non-increasing in $x \in \mathbb{R}$ when $c \leq d$.

The proof of (60) for the general case can now be completed as follows. We can find sequences of functions $\phi^{(n)}, \theta^{(n)}$ of the form (61) such that as $n \to \infty$, we have

$$
\int_{0}^{1} |\phi(s) - \phi^{(n)}(s)| \, ds \to 0, \quad \text{and} \quad \int_{0}^{1} |\theta(s) - \theta^{(n)}(s)| \, ds \to 0.
$$

(65)

Now for any two integrable functions $f$ and $g$ defined on $[0, 1]$ and for any $a \in \mathbb{R}, \varepsilon > 0$ we have

$$
\mu(\{s | f(s) \geq a, g(s) \leq a - \varepsilon\}) \leq \frac{1}{\varepsilon} \int_{0}^{1} |f(s) - g(s)| \, ds.
$$

(66)

This implies that as $n \to \infty$, we have

$$
\int_{0}^{1} |\bar{\phi}(s) - \bar{\phi}^{(n)}(s)| \, ds \to 0, \quad \text{and} \quad \int_{0}^{1} |\bar{\theta}(s) - \bar{\theta}^{(n)}(s)| \, ds \to 0,
$$

(67)

and then the result follows in a few lines.
The following is an easy consequence.

**Corollary 3.2**
Let $f$ satisfy the properties that were required in connection with (49). Then
\[
\int_{t_1}^{t_2} f(t + \bar{\theta}(t_1 + t_2 - t)) \, dt \leq \int_{t_1}^{t_2} f(t + \theta(t)) \, dt \leq \int_{t_1}^{t_2} f(t + \bar{\theta}(t)) \, dt \tag{68}
\]
when $t_1 + \bar{\theta}(t_1) \geq 0$, and the inequality signs are reversed when $t_2 + \bar{\theta}(t_2) \leq 0$.

**Proposition 3.3**
Let $V > 0$ and assume $\Theta_\nu \neq \emptyset$. Then there is a $\theta \in \Theta_\nu$ such that $J_\phi(\theta) \geq J_\phi(\eta)$ for all $\eta \in \Theta_\nu$.

**Proof**
Let $M = \sup \{J_\phi(\eta) \mid \eta \in \Theta_\nu\}$, and let $(\theta^{(k)})_{k \in \mathbb{N}}$ be a sequence in $\Theta_\nu$ such that $J_\phi(\theta^{(k)}) \to M$, as $k \to \infty$. We can write
\[
\theta^{(k)}(s) = \theta_+^{(k)}(s) - \theta_-^{(k)}(s), \; \text{ for all } s \in [t_1, t_2], \tag{69}
\]
where $\theta_\pm^{(k)}(s)$ are non-decreasing in $s$, and
\[
\theta_+^{(k)}(t_2) - \theta_+(t_1) + \theta_-^{(k)}(t_2) - \theta_-^{(k)}(t_1) \leq V. \tag{70}
\]
By Helly's theorem (see ref. 8, Sec. 11.2) we can find subsequences (again denoted by $\theta^{(k)}$, etc.) and non-decreasing functions $\theta_\pm(s)$ such that $\theta_\pm^{(k)}(s) \to \theta_\pm(s)$ in all but countably many points $s \in [t_1, t_2]$. At the same time it can be arranged that
\[
\theta_+(t_2) - \theta_+(t_1) + \theta_-(t_2) - \theta_-(t_1) \leq V, \tag{71}
\]
just by taking care that $t_1$ and $t_2$ are among the points $s$ with $\theta_\pm^{(k)}(s) \to \theta_\pm(s)$. When we let $\theta(s) = \theta_+(s) - \theta_-(s)$, we thus see that $\text{Var} (\theta; t_1, t_2) \leq V$, and, by dominated convergence, that
\[
\int_{t_1}^{t_2} f[\phi(s) + \theta(s)] \, ds = M. \tag{72}
\]
Finally, for any $a, b \in \mathbb{R}$, we have that
\[
\mu(\{s \mid a < \theta^{(k)}(s) < b\}) \to \mu(\{s \mid a < \theta(s) < b\}) \tag{73}
\]
when $k \to \infty$, again by dominated convergence. Since the numbers at the left hand side of (73) are all equal to $\mu(\{s \mid a < \bar{\theta}(s) < b\})$, we thus see that $\theta \in \Theta_\nu$, as required.

\[\square\]

The two following results, whose proofs are omitted, can be shown to hold by employing the same sort of arguments that were used to prove Proposition 3.1.

**Proposition 3.4**
We have
\[
\sup_{\theta \in \Theta} \int_{t_1}^{t_2} f[\delta(s) + \theta(s)] \, ds = \sup_{\phi \in \Phi} \int_{t_1}^{t_2} f[\phi(s) + \theta(s)] \, ds. \tag{74}
\]
We now return to the case that \( \phi(s) = s \) for all \( s \) (for convenience only).

**Proposition 3.5**
Let \( n \in \mathbb{N} \), and let \( \delta, s_l^{(n)}, \delta_l^{(n)} \) and \( \theta_l^{(n)} \) be defined as in (51), (52) and (53). Then we have
\[
\sup_{\theta \in \Theta} \int_{t_1}^{t_2} f[s + \theta(s)] \, ds = \lim_{n \to \infty} J(\theta_l^{(n)}). \tag{75}
\]
Note that in this last proposition there does not have to be a \( \theta \) such that \( J(\theta) = \lim_{n \to \infty} J(\theta_l^{(n)}) \). Although Proposition 3.4 shows that within the set \( \Theta \) there is a maximizer, say \( \theta_l \), it may well happen that \( \text{Var}(\theta_l; t_1, t_2) \to \infty \) as \( V \to \infty \). See also Example 5.5.

From Proposition 3.5 it follows that the continuous problem has a well-behaved solution when
\[
\max_{s} \bar{\theta}'(s) < 2. \tag{76}
\]
To see this, note that in this case we have for all \( n \in \mathbb{N} \)
\[
\max_{1 \leq i \leq n-1} (\delta_l^{(n)} - \delta_{l+1}^{(n)}) < \min_{1 \leq i \leq n-2} (s_i^{(n)} - s_{i+2}^{(n)}), \tag{77}
\]
see (35), whence the optimal \( \theta_l^{(n)} \) are all V-shaped. It thus follows that the stepfunctions \( \theta_l^{(n)} \) have uniformly bounded variation, they are asymptotically equimeasurable with \( \theta \), and
\[
\lim_{n \to \infty} J(\theta_l^{(n)}) = \sup_{\theta \in \Theta} J(\theta). \tag{78}
\]
Now proceed as in the proof of Proposition 3.1 to conclude the existence of a maximizer \( \theta \in \Theta \). Summarizing, we get the following.
Corollary 3.6
If \( \max_s \, \Theta'(s) < 2 \), then a V-shaped maximizer exists.

3.2. Maximizers with finite variation

In this subsection we assume that \( \phi(s) = s \) for all \( s \), and that we have a \( \Theta \in \Theta \), of finite variation such that \( J(\Theta) \geq J(\eta) \) for all \( \eta \in \Theta \) of finite variation. As we see from Proposition 3.1 this may occur without condition (76) being satisfied. Such a \( \theta \) is continuous at all but at most countably many points. We can redefine \( \theta \) so that it is continuous from the right on \([t_1, t_2]\) and continuous from the left at \( t_2 \), without violating the condition of being equimeasurable with \( \Theta \) or changing the value \( J(\Theta) \) of the functional. We start by establishing versions of Propositions 2.1 and 2.2 for the present case.

Proposition 3.7
Let \( t_1 < u < v < t_2 \). Then we have
(a) \( \theta(u) < \theta(v) \Rightarrow -[\theta(u) + \theta(v)] \leq u + v \),
(b) \( \theta(u) > \theta(v) \Rightarrow -[\theta(u) + \theta(v)] \geq u + v \)

Proof
(a) Suppose that \( \theta(u) < \theta(v) \), and that \( -[\theta(u) + \theta(v)] > u + v \). We can find two non-overlapping closed intervals \( I_u, I_v \) of equal length contained in \([t_1, t_2]\) such that \( u \in I_u, v \in I_v \) and such that for all \( s \in I_u \) and \( t \in I_v \), we have
\[
\theta(s) < \theta(t) \quad \text{and} \quad -[\theta(s) + \theta(t)] > s + t. \quad (79)
\]
Denoting \( I_u = [a, a + \delta] \) and \( I_v = [b, b + \delta] \), we have
\[
\int_{I_u} f[s + \theta(s)] \, ds + \int_{I_v} f[s + \theta(s)] \, ds
\]
\[
= \int_0^\delta f[a + x + \theta(a + x)] + f[b + x + \theta(b + x)] \, dx. \quad (80)
\]
Now because of (79), compare with the proof of Proposition 2.1,
\[
f[a + x + \theta(a + x)] + f[b + x + \theta(b + x)]
\]
\[
< f[a + x + \theta(b + x)] + f[b + x + \theta(a + x)] \quad (81)
\]
for all \( x \in (0, \delta) \). Hence
\[
\int_0^\delta f[a + x + \theta(a + x)] + f[b + x + \theta(b + x)] \, dx
\]
\[
< \int_0^\delta f[a + x + \theta(b + x)] + f[b + x + \theta(a + x)] \, dx. \quad (82)
\]
This shows that $\theta$ is not a maximizer since interchanging the values of $\theta$ on the intervals $I_u$ and $I_v$ increases the functional $J$. Contradiction.

(b) is proved similarly. \(\square\)

**Proposition 3.8**

Let $t_1 \leq u < v < w \leq t_2$. Then we have

(a) $\theta(u) > \theta(w) > \theta(v) \Rightarrow \theta(u) - \theta(v) \leq v - u$,

(b) $\theta(v) > \theta(u) > \theta(w) \Rightarrow \theta(u) - \theta(w) \geq w - u$,

(c) $\theta(v) > \theta(w) > \theta(u)$ does not occur.

**Proof**

The proof is the same as that of Proposition 2.2. \(\square\)

We are now ready to prove the following result.

**Theorem 3.9**

If the maximizer $\theta$ is of finite variation, then it is V-shaped.

**Proof**

First note that it suffices to prove the following:

Let $t_1 \leq u < v < w \leq t_2$. Then we have

$$\theta(v) > \theta(u) \Rightarrow \theta(w) \geq \theta(v). \tag{83}$$

To prove (83), suppose that $\theta(v) > \theta(u)$ and $\theta(w) < \theta(v)$. Let

$$\theta_0 := \inf \{\theta(x) | x < v\}. \tag{84}$$

Since $\theta$ is right-continuous at $v$, there is a $\delta > 0$ such that

$$\theta(z) > \theta_0, \text{ for all } z \text{ with } 0 \leq z - v < \delta. \tag{85}$$

We shall show that $\theta(w) \leq \theta_0 - \delta$. This implies that $\theta$, and therefore $\bar{\theta}$, does not assume values between $\theta_0 - \delta$ and $\theta_0$, which contradicts smoothness of $\bar{\theta}$.

To show that $\theta(w) \leq \theta_0 - \delta$, we let $\varepsilon > 0$ and $y < v$ be such that $\theta(y) < \theta_0 + \varepsilon < \theta(v)$. By Propositions 3.8(b) and 3.8(c) we see that

$$\theta(w) \leq \theta(y) - (w - y) < \theta_0 + \varepsilon - (w - v). \tag{86}$$

Hence, by letting $\varepsilon \downarrow 0$, we get that

$$\theta(w) \leq \theta_0 - (w - v). \tag{87}$$

From (85) it then follows that $w - v \geq \delta$, so that indeed $\theta(w) \leq \theta_0 - \delta$, which completes the proof. \(\square\)
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So, the optimal \( \theta \) is \( V \)-shaped, and consequently there is a \( v_0 \in [t_1, t_2] \) such that \( \theta(v_0) = \bar{\theta}(t_1) \) or \( \lim_{t \to v_0} \theta(t) = \bar{\theta}(t_1) \). The following theorem proves continuity of \( \theta \) on the left leg of the \( V \), provided that there is a left leg.

**Theorem 3.10**

If \( v_0 > t_1 \), then \( \theta \) is continuous on \([t_1, v_0)\).

**Proof**

Let \( u \in (t_1, v_0) \) be such that

\[
\theta(u) < \lim_{t \to u} \theta(t) \tag{88}
\]

By Theorem 3.9 and by smoothness of \( \bar{\theta} \) there is a \( v \geq v_0 \), such that

\[
\theta(y) \geq \lim_{t \to u} \theta(t) > \theta(v) > \theta(u), \text{ for all } y \text{ such that } t_1 \leq y < u. \tag{89}
\]

But then by Proposition 3.8(a) we get that

\[
0 \leq \theta(y) - \theta(u) \leq y - u, \text{ for all } y \text{ such that } t_1 \leq y < u, \tag{90}
\]

showing \( \lim_{t \to u} \theta(t) = \theta(u) \), a contradiction. Hence \( \theta \) is continuous on \((t_1, v_0)\).

Since \( \theta \) is right-continuous at \( t_1 \), the proof is complete. \( \square \)

We continue the analysis of \( \theta \) by studying its behaviour on the right leg of the \( V \). In order to simplify the analysis somewhat, we assume that \( \bar{\theta} \) is strictly increasing on \([t_1, t_2]\). As a consequence, \( \theta \) is strictly decreasing on \([t_1, v_0)\) and strictly increasing on \([v_0, t_2]\). The following result then follows immediately from Proposition 3.7.

**Corollary 3.11**

Let \( \bar{\theta} \) be strictly increasing on \([t_1, t_2]\) and let \( u \) and \( v \) with \( t_1 \leq u < v \leq t_2 \) be such that \( \theta(u) = \theta(v) \). Then we have \( \theta(u) = \theta(v) = -\frac{1}{2}(u + v) \) whenever \( u > t_1 \) or when \( \theta \) is continuous at \( v \).

From now on, we assume that \( t_1 < v_0 < t_2 \); other cases are trivial. In order to formulate the forthcoming results, it is necessary to introduce some more notations.

Firstly, let

\[
\bar{\varphi} := \begin{cases} 
\inf \{ t \in [v_0, t_2] | \theta(t) \geq \theta(t_1) \}, & \text{if } \theta(t_2) > \theta(t_1) \\
\frac{1}{2}v_0, & \text{otherwise.}
\end{cases} \tag{91}
\]

So, if \( \theta(t_2) > \theta(t_1) \), then we have \( \theta(\bar{\varphi}) = \theta(v_1) \) and it follows that \( \theta(v) = \bar{\theta}(v) \).
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Fig. 10. The graph of a V-shaped $\theta$, illustrating some definitions.

for all $v > \bar{v}$. Now, let

$$\theta_+ := \theta(\bar{v}), \quad \theta_- := \lim_{v \to \bar{v}} \theta(v),$$

(92)

and let $\hat{u}_+$ and $\hat{u}_-$ be the unique solutions $u \in [t_1, v_0]$ of the equations $\theta(u) = \theta_+$ and $\theta(u) = \theta_-$, respectively. (So $\hat{u}_+ = t_1$ when $\theta(t_2) > \theta(t_1)$.)

Secondly, set

$$\theta_{0,+} := \theta(v_0), \quad \theta_{0,-} := \lim_{v \to v_0} \theta(v) = \theta(t_1),$$

(93)

let $u_{0,+}$ be the unique solution $u \in [t_1, v_0]$ of the equation $\theta(u) = \theta_{0,+}$ and let $u_{0,-} = v_0$.

Finally, let $(v_k)_{k \geq 1}$ be an enumeration of the discontinuities of $\theta$ in $(v_0, \bar{v})$, let for all $k \geq 1$

$$\theta_{k,+} := \theta(v_k), \quad \theta_{k,-} := \lim_{v \to v_k} \theta(v),$$

(94)

and let $u_{k,+}$ and $u_{k,-}$ be the unique solutions $u \in [t_1, v_0]$ of the equations $\theta(u) = \theta_{k,+}$ and $\theta(u) = \theta_{k,-}$, respectively.

In fig. 10 we have plotted a case where $\theta$ is discontinuous at $\bar{v} < t_2$, at $v_0$ and at two other points $v_1, v_2 \in (v_0, \bar{v})$. Of course, all sorts of degeneracies can occur in the definitions of $\bar{v}, \hat{u}_\pm$, $v_{k,\pm}$, etc.

Theorem 3.12

We have

$$-\theta_{k,-} = \frac{1}{2}(u_{k,-} + v_k), \quad \text{for all } k \geq 1,$$

(95)
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\[ -\theta_{k,+} = \frac{1}{2}(u_{k,+} + v_k), \quad \text{for all } k \geq 1, \quad (96) \]
\[ -\theta_{0,+} = \frac{1}{2}(u_{0,+} + v_0), \quad (97) \]
\[ -\theta_- = \frac{1}{2}(\hat{u}_- + \hat{v}). \quad (98) \]

Furthermore, we have \( -\theta_{0,-} \geq v_0 \geq -\frac{1}{2}(\theta_{0,-} + \theta_{0,+}) \), and, if \( \theta(t_2) > \theta(t_1) \), then
\[ -\theta(t_1) = -\theta(\hat{v}) \leq \frac{1}{2}(t_1 + \hat{v}). \quad (99) \]

**Proof**
First we prove (99). This inequality follows from Proposition 3.7(b) by taking \( v \downarrow \hat{v} \) in the inequality
\[ -[\theta(t_1) + \theta(v)] \leq t_1 + v. \quad (100) \]
Similarly, one proves that \( -\theta_{0,-} \geq v_0 \geq -\frac{1}{2}(\theta_{0,-} + \theta_{0,+}) \).

Now (95) and (96) follow immediately from Corollary 3.11. This leaves us to show that
\[ -\theta_- = \frac{1}{2}(\hat{u}_- + \hat{v}), \quad (101) \]
because (97) is proved similarly. Let \( \epsilon > 0 \) and take a \( u \in (\hat{u}_-, \hat{u}_- + \epsilon) \) such that \( \hat{\theta}_- - \epsilon < \theta(u) < \hat{\theta}_- \). Next, take a \( v \in (\hat{\theta} - \epsilon, \hat{\theta}) \) such that \( \theta(u) < \theta(v) < \theta(\hat{\theta}) \). Then Proposition 3.7 shows that
\[ -\hat{\theta}_- > -\frac{1}{2}[\hat{\theta}_- + \theta(v)] - \frac{\epsilon}{2} \geq \frac{1}{2}(\hat{u}_- + \hat{v}) - \frac{\epsilon}{2} > \frac{1}{2}(\hat{u}_- + \hat{v}) - \epsilon, \quad (102) \]
and
\[ -\hat{\theta}_- < -\frac{1}{2}[\theta(u) + \theta(v)] \leq \frac{1}{2}(u + v) < \frac{1}{2}(\hat{u}_- + \hat{v}) - \frac{\epsilon}{2}. \quad (103) \]
Now let \( \epsilon \downarrow 0 \) to obtain (101).

Note that if \( t_1 < v_0 < t_2 \) and if \( \theta \) is continuous, then it follows that \( v_0 = -\theta(t_1) \). Another consequence of this theorem is the following.

**Corollary 3.13**
For each \( k \geq 1 \) there is at least one \( u \in (u_{k,+}, u_{k,-}) \) such that \( \theta'(u) = -\frac{1}{2} \). Consequently, the number of discontinuities of \( \theta \) is finite whenever the number of points \( t \in [t_1, t_2] \) with \( \partial'(t) = \frac{1}{2} \) is finite.
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Fig. 11. Illustration of the intervals \((u_i, u_p)\) and \((v_i, v_p)\) of Proposition 3.14.

Proof
Note that \(\theta\) is smooth on each interval \((u_{k-}, u_{k+})\), and that

\[
\theta(u_{k-}) - \theta(u_{k+}) = -\frac{1}{2}(u_{k-} - u_{k+}),
\]

so that the result follows from the mean value theorem.

Now that we know the conditions on the boundaries of the intervals into which \([t_1, t_2]\) is divided, we will describe \(\theta\) in more detail in terms of \(\bar{\theta}\) on each of these intervals. From now on, assume that the number of points \(t \in [t_1, t_2]\) with \(\theta'(t) = \frac{1}{2}\) is finite, so that we have discontinuities at, say, \(v_1, \ldots, v_K\), with \(v_1 < v_2 < \cdots < v_K\). (Here \(v_K\) may or may not be equal to \(\hat{o}\), if \(\hat{o} < t_2\).)

It is straightforward to express \(\theta\) on the intervals

\((u_{k-}, u_{k+})\), \((u_0-, u_0+)\), \((\hat{o}, t_2)\), \((\hat{u}+, \hat{u}-)\) or \((t_1, \hat{u}+)\)

in terms of \(\bar{\theta}\), and so is their contribution to \(J(\theta)\). This is not true for the remaining intervals \((u_{k+1-}, u_{k+})\) and \((v_k, v_{k+1})\), which will be treated as pairs, and for which the following result is relevant.

Proposition 3.14
Let \(U = (u_i, u_p)\), \(V = (v_i, v_p)\) be one of the above mentioned pairs of intervals, and let \(S = (s_2, s_p)\), where \(s_2\) and \(s_p\) are the solutions \(s\) of

\[
\bar{\theta}(s) = \theta(v_i)(= \theta(u_p))\]

and \(\bar{\theta}(s) = \theta(v_p)(= \theta(u_i))\), respectively. Define functions \(u, v: S \to U, V\) by (see fig. 11)

\[
\theta(u(s)) = \theta(v(s)) = \bar{\theta}(s), \quad \text{for all } s \in S.
\]

Then we have for all \(s \in S\)

\[
u(s) = -\frac{1}{2}(s - t_1) - \bar{\theta}(s), \quad v(s) = \frac{1}{2}(s - t_1) - \bar{\theta}(s).
\]

Furthermore, for all \(s \in S\) we have \(\theta'(s) \leq \frac{1}{2}\), and

\[
\theta'[u(s)] = \frac{-\bar{\theta}'(s)}{\frac{1}{2} + \bar{\theta}'(s)} \in [-\frac{1}{2}, 0],
\]
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and

$$
\theta'(v(s)) = \frac{\bar{\theta}'(s)}{\frac{1}{2} - \bar{\theta}'(s)} \in [0, \infty].
$$

(109)

Finally,

$$
\int_{\mathcal{U}} f[u + \theta(u)] du + \int_{\mathcal{V}} f[v + \theta(v)] dv = -2 \int_S f[\frac{1}{2}(s - t_1)] \bar{\theta}'(s) ds.
$$

(110)

Proof

We have for any two points $u \in \mathcal{U}$ and $v \in \mathcal{V}$ with $\theta(u) = \theta(v)$ that

$$
-\theta(u) = -\theta(v) = \frac{1}{2}(u + v).
$$

(111)

Hence $u(s)$ and $v(s)$ of (106) satisfy

$$
u(s) + v(s) = -2\bar{\theta}(s),
$$

(112)

for all $s$. Furthermore, since $[u(s), v(s)] = \{u|\theta(u) \leq \bar{\theta}(s)\}$, we have

$$
v(s) - u(s) = s - t_1.
$$

(113)

Now (107) follows from these two equalities.

To show $\bar{\theta}'(s) \leq \frac{1}{2}$, we note that $v(s)$ is non-decreasing and is related to $\bar{\theta}(s)$ by the second formula in (107).

To show (108) and (109), we note that

$$
\theta'(u(s)) = \frac{1}{u'(s)} \frac{d}{ds} [\theta(u(s))] = \frac{-\bar{\theta}'(s)}{\frac{1}{2} + \bar{\theta}'(s)} \in [-\frac{1}{2}, 0],
$$

(114)

and

$$
\theta'(v(s)) = \frac{1}{v'(s)} \frac{d}{ds} [\theta(v(s))] = \frac{\bar{\theta}'(s)}{\frac{1}{2} - \bar{\theta}'(s)} \in [0, \infty].
$$

(115)

Finally, to show (110), we note that by the substitutions $u = u(s), v = v(s)$, we get

$$
\int_{\mathcal{U}} f[u + \theta(u)] du = -\int_S f[u(s) + \bar{\theta}(s)] u'(s) ds,
$$

(116)

and

$$
\int_{\mathcal{V}} f[v + \theta(v)] dv = \int_S f[v(s) + \bar{\theta}(s)] v'(s) ds,
$$

(117)

respectively. Adding these two equalities, and using (107) and oddness of $f$ we get the required result.  

A further result is the following. Suppose there is an interval \((u_{k,+}, u_{k,-})\) as above. Then \(\theta\) has at least two points \(s\) where \(\theta'(s) = \frac{1}{2}\). This is seen as follows. From (104) it follows that there are \(s_i\) and \(s_p\) such that

\[
\theta(s_i) - \theta(s_p) = \frac{1}{2}(s_i - s_p); \quad \text{and} \quad \theta'(s_i), \theta'(s_p) \leq \frac{1}{2},
\]

where for the last two inequalities Proposition 3.14 has been used. Therefore, in the cases that \(\theta'(s) = \frac{1}{2}\) for only one \(s\), there is at most one interval to which the analysis of Proposition 3.14 applies. These cases are worked out further in the next section.

4. Analytic results for the continuous case

In the previous section we have indicated how we can express \(J(\theta)\), in the case of finitely many points \(s\) with \(\theta'(s) = \frac{1}{2}\) and under the assumption that the optimizers are of finite variation, as a series of integrals involving \(f, \theta\) and with integration bounds (determined by the discontinuities of \(\theta\)) satisfying certain constraints. Hence the problem has been reduced to a finite-dimensional constrained optimization problem. However, this problem can get quite involved since the constraints are not so easy to deal with. In this section we present analytic results for the case that \(\theta'(s) = \frac{1}{2}\) for at most one \(s \in [t_1, t_2]\), again under the assumption that the optimizers are of finite variation (which is for instance true if \(\theta'(s) < 2\) for all \(s\)). This already shows that analytic solution of the problem quickly gets cumbersome.

4.1. The case \(\theta'(s) > \frac{1}{2}\) for all \(s\)

Suppose that \(\theta'(s) > \frac{1}{2}\) for all \(s \in [t_1, t_2]\). Then \(\theta\) can have at most one discontinuity: at \(v_0\). Furthermore, \(\theta\) must be injective, otherwise there would be two intervals to which the analysis of Proposition 3.14 applies. This would imply that \(\theta'(s) \leq \frac{1}{2}\) on a certain interval, contradicting our assumption on \(\theta\). Therefore, the optimal \(\theta\) is of the form as depicted in fig. 12; where the degenerate cases \(v_0 = t_1\) or \(v_0 = t_2\) may also occur. From now on, we will write \(v\) instead of \(v_0\). The value of \(J(\theta)\) is given by

\[
\psi(v) := \int_{t_1}^{v} f[t_1 + v - s + \theta(s)] ds + \int_{v}^{t_2} f[s + \theta(s)] ds\]

(119)

where \(v\) is constrained so as to satisfy (see Theorem 3.12)

\[
-\theta(v) \leq \frac{1}{2}(t_1 + v),
\]

(120)

unless \(v = t_1\) or \(v = t_2\). Hence we should maximize \(\psi(v_0)\) over \(v_0 \in [t_1, t_2]\) satisfying (120).
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Fig. 12. The optimal $\theta$ for the case that $\partial'(\alpha) > \frac{1}{2}$ for all $s$.  

We shall show that $\psi''(v) < 0$ whenever $v \in (t_1, t_2)$ and $-\partial(v) \leq \frac{1}{2}(t_1 + v)$. To that end we note that we have

$$\psi'(v) = f[t_1 + \partial(v)] - f[v + \partial(v)] + \int_{t_1}^{v} f'[\tau + t_1 + v + \partial(s)] ds$$  \hspace{1cm} \text{(121)}$$

and

$$\psi''(v) = \int_{t_1}^{v} f''[t_1 + v - s + \partial(s)] ds + [1 + \partial'(v)] \{ f'[t_1 + \partial(v)] - f'[v + \partial(v)] \}.$$  \hspace{1cm} \text{(122)}$$

So

$$\psi''(v) = \int_{t_1}^{v} f''[t_1 + v - s + \partial(s)] ds - [1 + \partial'(v)] \int_{t_1}^{v} f''[t_1 + v - s + \partial(v)] ds.$$  \hspace{1cm} \text{(123)}$$

Now, by (120) and the fact that $f''$ is increasing,

$$\int_{t_1}^{v} f''[t_1 + v - s + \partial(v)] ds \geq \int_{t_1}^{v} f''[\frac{1}{2}(t_1 + v) - s] ds = 0.$$  \hspace{1cm} \text{(124)}$$

Therefore, since $\partial'(v) \geq 0$, we have

$$\psi''(v) \leq \int_{t_1}^{v} f''[t_1 + v - s + \partial(s)] - f''[t_1 + v - s + \partial(v)] dv.$$  \hspace{1cm} \text{(125)}$$

Finally, $\partial(s) < \partial(v)$ for $s \in [t_1, v)$ and $f''$ is increasing, whence $\psi''(v) < 0$, as required.
Fig. 13. In the case that \( \bar{\theta}'(s) < \frac{1}{2} \) for all \( s \), discontinuities at \( v_0 \) will not occur.

It follows that the optimal \( \theta \) has \( v = t_1 \) or \( v = t_2 \), unless there is a \( v \in (t_1, t_2) \) with \( \psi'(v) = 0 \). (To see whether there is such a \( v \), first solve \(-\bar{\theta}(w) = \frac{1}{2}(t_1 + w)\). Then for this \( w \) we have \( \psi'(w) > 0 \). Now also check whether \( \psi'(t_2) < 0 \).)

4.2. The case \( \bar{\theta}'(s) < \frac{1}{2} \) for all \( s \)

Suppose that \( \bar{\theta}'(s) < \frac{1}{2} \) for all \( s \in [t_1, t_2] \). Note that in this case the condition of the optimizer being of finite variation is automatically fulfilled. Then, as in the previous subsection, \( \theta \) can have at most one discontinuity: at \( v_0 \). We will show that \( \theta \) has no discontinuity at all. To this end, consider the situations depicted in figs 13(a) and 13(b). We will show that none of these situations can occur, so that \( \theta \) must be of the form as depicted in figs 14(a) or 14(b), where, again, the degenerate cases \( v_0 = t_1 \) or \( t_2 \) may occur. We will consider the situation of fig. 13(a) only; the other case is treated completely similarly.

Let us introduce some notation. Let, as in Sec. 3, \( u_{0,+} \) be such that \( \theta(u_{0,+}) = \theta(v_0) \). Also, let \( s_1 \) be such that \( \bar{\theta}(s_1) \) that \( \theta(s_1) = \theta(v_0) \), and let \( s_p \) be such that \( \bar{\theta}(s_p) = \theta(t_1) \). Then Proposition 3.14 applies to the intervals \( (u_{0,+}, u_p) : = (t_1, u_{0,+}) \) and \( (v_2, v_p) : = (v_0, \bar{\theta}) \). It follows from Proposition 3.14 that then \( J(\theta) \) can be written as

\[
\psi(s_1) = C - 2 \int_{s_1}^{s_p} f[\frac{1}{2}(s - t_1)]\bar{\theta}'(s) \, ds + \int_{t_1}^{s_1} f[t_1 + v(s_1) - u + \bar{\theta}(u)] \, du,
\]

where \( C \) does not depend on \( s_1 \), and where \( v \) is the function (see (107)) defined

Fig. 14. Two possibilities for the optimal \( \theta \) for the case that \( \bar{\theta}'(s) < \frac{1}{2} \) for all \( s \).
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by

\[ v(s) = \frac{1}{2}(s - t_i) - \bar{\theta}(s). \]  

(127)

We will show that \( \psi'(s_i) < 0 \) for all \( s_i < s_p \), so that \( s_i = t_i \) yields the maximum value for (126), which implies that the optimal \( \theta \) is continuous at \( v_0 \).

**Lemma 4.1**

We have \( \psi'(s_i) < 0 \).

**Proof**

Differentiating (126) gives

\[
\psi'(s_i) = \int_{t_i}^{s_i} f'[t_i + v(s_i) - u + \bar{\theta}(u)]\psi'(s_i) \, du + 2f[\frac{1}{2}(s_i - t_i)]\bar{\theta}'(s_i) + f[t_i + v(s_i) + \theta(s_i)].
\]

(128)

It follows from the definition of \( v \) and from the oddness of \( f \) that this equals

\[
\psi'(s_i) = -\left[\frac{1}{2} - \bar{\theta}'(s_i)\right] \left\{2f[\frac{1}{2}(s_i - t_i)] - \int_{t_i}^{s_i} f'[t_i + v(s_i) - u + \bar{\theta}(u)] \, du\right\}.
\]

(129)

We will show that the expression in (129) between braces is positive. Define for all \( u \in [t_i, s_i] \)

\[
\omega(u) := -[t_i + v(s_i) - u + \bar{\theta}(u)]
\]

(130)

and let

\[
a := \omega(t_i); \quad b := \frac{1}{2}(s_i - t_i) = \omega(s_i).
\]

(131)

Then, since \( f' \) is even, we get by substituting \( x = \omega(u) \),

\[
\int_{t_i}^{s_i} f'[t_i + v(s_i) - u + \bar{\theta}(u)] \, du = \int_a^b f'(x)h(x) \, dx,
\]

(132)

where \( h(x) = [\omega'(\omega^{-1}(x))]^{-1} \). Since \( \frac{1}{2} < \omega'(u) < 1 \), we have \(-b < a < 0\) and \(1 < h(x) < 2\) for \( x \in [a, b] \). Also

\[
\int_a^b h(x) \, dx = 2b.
\]

(133)

From the properties of \( f' \) (even, increasing on \([0, \infty)\)) it then follows that

\[
\int_a^b f'(x)h(x) \, dx < \int_{-b}^b f'(x) \, dx = 2f(b) = 2f[\frac{1}{2}(s_i - t_i)],
\]

(134)
Fig. 15. Two possibilities for the optimal $\theta$ for the case that $\vartheta'(s) < \frac{1}{2}$ for $s < \bar{s}, \vartheta'(s) > \frac{1}{2}$ for $s > \bar{s}$.

as required.

So the optimal $\theta$ is of the form as depicted in figs 14(a) and 14(b), where the degenerate cases $v_0 = t_1$ or $v_0 = t_2$ and $\vartheta = t_2$ or $\hat{u} = t_1$ should be expected to occur. Note that $v_0 = -\vartheta(t_1)$.

The points $\vartheta, \hat{u}$ are found by solving $v, u$ from

\[-\vartheta(v) = \frac{1}{2}(t_1 + v), \quad -\vartheta(t_1 + t_2 - u) = \frac{1}{2}(u + t_2), \quad (135)\]

respectively. At most one of these equalities has a solution. The precise situation can easily be read off from a picture of $\vartheta$: the first equation in (135) has a (unique) solution $\vartheta \in (t_1, t_2)$ if and only if

\[\vartheta(t_1) < -t_1 \quad \text{and} \quad \vartheta(t_2) > -\frac{1}{2}(t_1 + t_2), \quad (136)\]

and the second equation in (135) has a (unique) solution $\hat{u} \in (t_1, t_2)$ if and only if

\[\vartheta(t_1) > -t_2 \quad \text{and} \quad \vartheta(t_2) < -\frac{1}{2}(t_1 + t_2). \quad (137)\]

Note that at most one of the conditions (136) and (137) can be fulfilled. If none of these conditions is fulfilled, we have the following. If $\vartheta(t_2) = -\frac{1}{2}(t_1 + t_2)$, then we get the degenerate solution $\hat{u} = t_1$. If $\vartheta(t_2) > -\frac{1}{2}(t_1 + t_2)$ and $\vartheta(t_1) \geq -t_1$, then we get the degenerate solution $\vartheta = t_1$. Finally, if $\vartheta(t_2) < -\frac{1}{2}(t_1 + t_2)$ and $\vartheta(t_1) \leq -t_2$, then we get the degenerate solution $\hat{u} = t_1$.

4.3. The case of one point $\bar{s}$ with $\vartheta'(\bar{s}) = \frac{1}{2}$

We distinguish two subcases.

Case 1
Suppose $\vartheta'(s) < \frac{1}{2}$ for $s < \bar{s}$ and $\vartheta'(s) > \frac{1}{2}$ for $s > \bar{s}$. In this case the optimal $\theta$ has the form as depicted in figs 15(a) and 15(b) or several degeneracies thereof. For instance in fig. 15(a), for given $\vartheta$ the point $\hat{u}$ is determined as the
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Fig. 16. Two possibilities for the optimal $\theta$ for the case that $\theta'(s) > \frac{1}{2}$ for $s < \tilde{s}$, $\theta'(s) < \frac{1}{2}$ for $s > \tilde{s}$.

solution $u$ of

$$-\theta(t_1 + t_2 - \theta + u) = \frac{1}{2}(u + \theta),$$  \hspace{1cm} (138)

or $u_- = \theta$ when no solution exists. Then the corresponding value of the functional $J$ should be maximized as a function of $\theta$, using pretty much the same methods as in the previous subsections. We shall not do this here.

Case 2

Suppose $\theta'(s) > \frac{1}{2}$ for $s < \tilde{s}$ and $\theta'(s) < \frac{1}{2}$ for $s > \tilde{s}$. In this case the optimal $\theta$ has the form as depicted in figs 16(a) and 16(b) or several degeneracies thereof. For instance in fig. 16(a), the point $u_-$ is determined as the solution $u$ of

$$-\theta(t_1 + t_2 - u) = \frac{1}{2}(u + \theta).$$  \hspace{1cm} (139)

For determining the points $u_{0,+}$ and $v_0$, we should use the approach of Subsec. 4.2. To that end we consider for $s_i = \tilde{s}$, see (129),

$$\phi(s_i) := 2f\left[\frac{1}{2}(s_i - t_1)\right] - \int_{t_1}^{s_i} f'[t_2 + \psi(s_i) - u + \theta(u)]du. \hspace{1cm} (140)$$

As in (126)–(134) it can be shown that $\phi(\tilde{s}) < 0$, and that for $s > s_i$ we have

$$\phi'(s_i) = \left[\frac{1}{2} - \theta'(s_i)\right] \int_{t_1}^{s_i} f''[t_2 + \psi(s_i) - u + \theta(u)]du > 0. \hspace{1cm} (141)$$

It is then easy to see that $-\left[\frac{1}{2} - \theta'(s_i)\right]\phi(s_i)$ has at most one zero $s_i > \tilde{s}$, and that we should set $u_{0,+} = t_1 + v_0 - s_i$ if such an $s_i$ exists and $u_{0,+} = t_1 + v_0 - \tilde{s}$ otherwise.

5. Examples

In this section we present some examples that exhibit a number of features
we have shown the optimal \( \theta \) to possess, both for the discrete and the continuous maximization problem. It turns out that the cases of uniform reflection angle distribution (i.e. such that \( \vartheta'(t) \) or \( \vartheta_n - \vartheta_{n-1} \) is constant) already provide a good picture of the various phenomena. These cases can be treated analytically, and are of practical relevance to the reflector design problem.

**Example 5.1**

Let \( 0 < \alpha < \frac{1}{2} \), let \( b > 0 \), and consider the function \( \vartheta_a : [-b, b] \to \mathbb{R} \), defined by

\[
\vartheta_a(s) = \alpha(s - b). \tag{142}
\]

Considering (135), note that we have \( -\vartheta(b) = \frac{1}{2}(-b + b) \), so \( \theta = b = t_2 \). Furthermore, we have \( v_0 = -\vartheta(t_1) = 2ab \). Now (see Proposition 3.14), we have

\[
u(s) = -\frac{1}{2}(s - t_1) - \vartheta(s) = -(\frac{1}{2} + \alpha)s + (-\frac{1}{2} + \alpha)b, \tag{143}
\]

\[
v(s) = \frac{1}{2}(s - t_1) - \vartheta(s) = (\frac{1}{2} - \alpha)s + (\frac{1}{2} + \alpha)b. \tag{144}
\]

It easily follows from (106) that

\[
\begin{cases}
-\frac{\alpha}{\frac{1}{2} + \alpha}(u + b) & \text{for all } u \in [-b, 2ab], \\
\frac{\alpha}{\frac{1}{2} - \alpha}(u - b) & \text{for all } u \in [2ab, b].
\end{cases}
\]

For the functional we get

\[
J(\theta_a) = -2 \int_{-b}^{b} f[\frac{1}{2}(s + b)]a \, ds = -4\alpha \int_{0}^{b} f(u) \, du. \tag{146}
\]

Note that

\[
\lim_{\alpha \to 1} \theta_a(u) = -\frac{1}{2}(u + b) = \vartheta_1(-u), \tag{147}
\]

for all \( u \in [-b, b] \).

The more general case, where, instead of \( \theta_a \), we consider

\[
\vartheta_{a,\beta} = as + \beta \tag{148}
\]

with \( \beta \in \mathbb{R} \) and \( 0 < \alpha < \frac{1}{2} \), can be reduced to the one above, but we shall not work this out in detail here.
Example 5.2
Let \( 1 \neq \alpha > \frac{1}{2}, \beta \in \mathbb{R} \) and \( b > 0 \), and consider \( \vartheta_{a,b}^\alpha : [-b, b] \to \mathbb{R} \), defined by (148). In case \( \alpha \geq 2 \), it may happen that there does not exist a maximizer of finite variation; we come back to this point in Example 5.5. When we assume, however, the existence of such a maximizer, we can apply the analysis of Secs 3.2 and 4.1, and this we do below. Stated differently we determine the maximum of \( J(\theta) \) over all V-shaped \( \theta \) equimeasurable with \( \vartheta_{a,b}^\alpha \). When

\[
-\alpha b + \beta - b \geq 0 \quad \text{or} \quad \alpha b + \beta + b \leq 0,
\]

(149)

the maximizer is given as \( \theta(u) = \vartheta_{a,b}^\alpha (u) \) or \( \theta(u) = \vartheta_{a,b}^\alpha (-u) \). Otherwise, that is, when

\[
|\beta| < (1 + \alpha)b,
\]

(150)

we should look for a solution \( v \) of the equation

\[
-f[v + \vartheta(v)] + f[-b + \vartheta(v)] + \int_{-b}^{v} f'[v - b - s + \vartheta(s)] ds = 0,
\]

(151)

or, in the present case, of

\[
f(v - \alpha b + \beta) = \alpha f(\alpha v - b + \beta) + (1 - \alpha)f((1 + \alpha)v + \beta).
\]

(152)

Here \( v \) is constrained by

\[
-\vartheta(v) \leq \frac{1}{2}(v - b),
\]

(153)

so, in the present case, by

\[
v \geq \frac{1}{2}b - \beta \frac{1}{2} + \alpha.
\]

(154)

There is at most one such \( v \); if such a \( v \) exists we have \( v_0 = \min (v, b) \), and otherwise \( v_0 = b \), see fig. 12. Note that the "obvious" solution \( v = -b \) of (152) does not meet the constraint (154) when (150) is valid. Having found \( v_0 \), we get for the functional

\[
J = \int_{-b}^{v_0} f[(1 - \alpha)s + \beta + \alpha v_0 - \alpha b] ds + \int_{v_0}^{b} f[(1 + \alpha)s + \beta] ds.
\]

(155)

We note that the limiting case \( \alpha = \frac{1}{2}, \beta = -b/2 \) (see Example 5.1) has \( v = b \) as a solution of (152) satisfying the constraint (154) with equality. This agrees with (147).
Example 5.3
Let $\alpha = 1$, $\beta \in \mathbb{R}$ and consider $\vartheta_{1,\beta} : [-b, b] \to \mathbb{R}$, defined by
\[
\vartheta_{1,\beta}(s) = s + \beta. \quad (156)
\]
The analysis of this case is the same as in the previous example, except that (152) has to be replaced by
\[
f(2v + \beta) = f(v - b + \beta) + (v + b)f'(v - b + \beta). \quad (157)
\]
For instance, in the case $\beta = 0, f(s) = s^3$, eq. (152) reads $(b - 2v)(b + v)^2 = 0$, so we have $v_0 = b/2$, and we find
\[
J(\vartheta_{1,0}) = \frac{\pi}{4} b^4. \quad (158)
\]
Now before we consider a case with $\vartheta'(s) > 2$ for some $s$, we consider a discrete problem.

Example 5.4
Let $n = 2k + 1$, and let
\[
s_i := i - 1 - k, \quad \vartheta_i = \frac{1}{2}(i - n). \quad (159)
\]
Then it can be shown by Proposition 2.10 that the optimal $\theta$ is as plotted in fig. 17(a). When we exchange the roles of the $s_i$s and $\vartheta_i$s, i.e. if $\vartheta_i := i - 1 - k$ and $s_i = \frac{1}{2}(i - n)$, then the optimal $\theta$, depicted in fig. 17(b), is of course the inverse of the one above.

Example 5.5
Let $b > 0$ and consider $\vartheta_{4, -\frac{1}{4}} : [-b/4, b/4] \to \mathbb{R}$, defined by
\[
\vartheta(s) = \vartheta_{4, -\frac{1}{4}}(s) = 4s - \frac{1}{4}. \quad (160)
\]
We wish to calculate
\[
\sup_{\theta \in \Theta} \int_{-b/4}^{b/4} f(s + \theta(s)) \, ds. \quad (161)
\]
To that end, let $\tilde{\varphi} : [-b/4, b/4] \to \mathbb{R}$ be defined by
\[
\tilde{\varphi}(s) = s, \quad (162)
\]
and we get by Proposition 3.4 and Example 5.1 that
\[
\sup_{\theta \in \Theta} \int_{-b/4}^{b/4} f[s + \theta(s)] \, ds = \sup_{\phi \in \Phi} \int_{-b/4}^{b/4} f \left[ 4s + \phi(s) - \frac{b}{4} \right] \, ds = \frac{1}{4} \sup_{\phi \in \Phi} \int_{-b}^{b} f \left[ s + \phi \left( \frac{s}{4} \right) - \frac{b}{4} \right] \, ds = -\frac{1}{4} \int_{0}^{b} f(u) \, du. \quad (163)
\]
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Fig. 17. Graphs of the optimizers of Example 5.4 for $k = 7$.

Here, $\sup_\phi$ is assumed by $\phi$ given by

$$\phi(s/4) - b/4 = \theta_4(s);$$

see Example 5.1. For instance, in the special case that $b = 1, f(s) = s^3$, we find that the supremum in (163) equals $-0.0625$ while the value of (155), the supremum of $J(\theta)$ over all V-shaped $\theta_s$, equals $-0.0664$.

It should be noted that the optimal $\phi$ is not injective. Hence the measure preserving mapping $\pi = \tilde{\phi}^{-1} \circ \phi$ of $[-b, b]$ onto itself is not injective. This
explains why the supremum in (161) is not assumed: the only possible candidate would be \( \overline{\theta}(\pi^{-1}(s)) \), but this is not properly defined as a one-valued function. In fig. 18 we have plotted \( \theta_{1/4} \) on \([-b, b]\), \( \phi \) on \([-b/4, b/4]\), the two-valued inverse \( \phi^{-1} \) of \( \phi \) on \([-b/4, b/4]\), and the two-valued "optimal"

\[
\theta(s) = 4\pi^{-1}(s) - b/4 = 4\phi^{-1}(s) - b/4
\]

(165)
on \([-b/4, b/4]\). If one denotes by \( \theta_\nu \) an optimal element of \( \Theta_\nu \), see Proposition 3.3, one would observe that \( \theta_\nu \) goes back and forth between the two straight lines in fig. 18(d) at a rate that tends to infinity as \( V \to \infty \).
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