ON A FIRST-PASSAGE-TIME PROBLEM

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Summary

In this fluctuation problem we ask for the probability that a function starting at a time \( t = 0 \) with a value \( E_0 \) will never have passed a value \( E_1 \) in the time interval \((0, t_1)\). This probability is calculated (a) from the Fokker-Planck equation with suitable boundary conditions and (b) from an integral-equation derived by Schrödinger.

Résumé

On demande la probabilité qu'une fonction aléatoire stationnaire de Laplace-Gauss ne passe pas une valeur \( E_1 \) dans l'intervalle \((0, t_1)\) s'il est donné qu'à l'instant \( t = 0 \) cette valeur est \( E_0 \). Cette probabilité est calculée (a) de l'équation Fokker-Planck avec des conditions initiales bien choisies et (b) d'une équation intégrale due à Schrödinger.

Zusammenfassung

Die Wahrscheinlichkeit wird gefragt, daß eine zufällige Variable \( E(t) \) an dem Zeitpunkt \( t_1 \) das erste Mal den Wert \( E_1 \) durchschreitet wenn gegeben ist daß \( E(0) = E_0 \). Diese Wahrscheinlichkeit wird berechnet (a) aus der Fokker-Planckschen Gleichung mit geeigneten Randwertbedingungen und (b) aus einer Integral-Gleichung von Schrödinger.

1. Introduction

The following problem was put to us by Mr Hepp of this laboratory: A condenser is charged by pulses of equal amplitude, very short duration and with a Poisson distribution in time \(^*\). Parallel to the condenser is a resistor. After some time the condenser voltage will fluctuate about an equilibrium value \( E_0 \). Starting at a time \( t = 0 \) with a condenser voltage \( E_0 \), we ask for the probability that at a time \( t = t_1 > 0 \) the voltage will never have passed a value \( E_1 \). Since the decrease of this probability during a time \( \Delta t \) is the probability that the value \( E_1 \) has been passed during that interval for the first time, the solution of our problem and that of the first-passage-time problem are equivalent. The problem also arises when we charge the condenser by a current with shot-effect, though on a different time scale.

The circuit arrangement is given in fig. 1. If there is no charge on the condenser for \( t < 0 \), the effect of a single current pulse \( Q \delta(t) \) is a voltage on the condenser of amount

\(^*\) The Poisson distribution is characterized by the following assumptions \(^1\): (1) The probability of an event in any time interval \( dt \) is asymptotically \( vdt \). (2) The probability of more than one event in a time interval \( dt \) is of smaller order of magnitude than \( dt \). (3) The number of events in non-overlapping time intervals represent independent random variables.
\[ V(t) = \frac{Q}{C} e^{-t/RC} U(t), \]  
(1)

where \( \delta(t) \) is the Dirac \( \delta \)-function:

\[ \delta(t) = \begin{cases} 0, & t \neq 0, \\ \frac{1}{\pi t}, & t \to 0. \end{cases} \]

\[ \int_{-\infty}^{+\infty} \delta(t) \, dt = 1, \]

and \( U(t) \) is the unit function:

\[ U(t) = \begin{cases} 0, & t < 0; \quad U(0) = \frac{1}{2}; \quad U(t) = 1, & t > 1. \end{cases} \]

In the general theory of fluctuations and Brownian movement many interesting relations have been derived which can be applied to our problem. Presuming that the reader wishes to know what can be said about averages and probabilities in a case like this, we will first give a short survey of some important relations. For the proofs we refer to the original papers, in particular those of Rice \(^2\), Wang and Uhlenbeck \(^3\), Ornstein and Uhlenbeck \(^4\).

First we have Campbell’s theorem. This states that the average value \( E_0 \) of \( V(t) \) is

\[ E_0 = \overline{V(t)} = \frac{Q v}{C} \int_{-\infty}^{\infty} e^{-t/RC} U(t) \, dt = Q R v. \]

(2)

The mean-square value of the fluctuation about this average is

\[ \langle (V(t) - \overline{V(t)})^2 \rangle_{Av} = \frac{v Q^2}{C^2} \int_{-\infty}^{\infty} e^{-2t/RC} U(t) \, dt = \frac{Q^2 R v}{2C} = \frac{E_0^2}{2a^2}, \]

(3)

where \( a^2 = R C v. \)

Let the proportion of time that \( V(t) \) spends in the range \( V, V + dV \) be \( P(V) \, dV. \) Then the approach of \( P(V) \) to a normal law may be expressed by a series, known as Edgeworth’s series \(^2\). Introducing a new variable

\[ b = \frac{a(V - E_0) \sqrt{2}}{E_0}, \]
the first terms are
\[ P(b)db = \frac{1}{\sqrt{2\pi}} e^{-b^2/2} \left[ 1 - \frac{2^{1/3}(3b - b^3)}{9a} + \frac{1}{648a^2} (-39 + 198b^2 - 93b^4 + 8b^6) + \ldots \right] db. \]

We shall assume that for the values of \(a\) and \(b\) of interest in our problem the first term gives a sufficient approximation. Possible corrections will be referred to later on.

The correlation function is defined by
\[ \rho(\tau) = \frac{\langle b(t)b(t+\tau) \rangle}{b^2}, \]
which in our case leads to \( \rho(\tau) = e^{-\tau/RC}. \)

Its Fourier transform \( S(f) \) is called the normalized spectrum. Here \( S(f) = 4RC/(1 + 4\pi^2f^2RC^2). \)

It is known that if the correlation function is \( e^{-\tau/RC} \) the process is a Markoff process, i.e. the transitions from one voltage level to another in non-overlapping time intervals are statistically independent. Thus we can define a probability \( P(b_0,b,t)db \) that, given \( b_0 \), one finds \( b \) in the range \( b, b + db \) a time \( t \) later. This probability function satisfies a diffusion equation, the Fokker-Planck equation, which reads
\[ RC \frac{\partial P}{\partial t} = \frac{\partial}{\partial b} (bP) + \frac{\partial^2 P}{\partial b^2}. \]

Its fundamental solution is
\[ P(b_0,b,t) = \left( \frac{2\pi}{RC} \right)^{-1/2} e^{-b^2/RC} \left( 1 - e^{-2b_0^2/RC} \right)^{1/2}; \]
\[ P(b_1,t) = 0, \quad \tau > 0, \]
\[ P(+\infty,\tau) = 0 \quad \tau > 0. \]

2. Application of the Fokker-Planck equation to our problem

For our first-passage problem we are only interested in those solutions of the Fokker-Planck equation that have not passed a certain value \(-b_1\).

A separation of variables can be effected by the substitution of \( P = e^{-\lambda F(b)} \), where \( \lambda \) is, for the present, an unspecified constant. Equation (5) now leads to the differential equation
\[ \frac{d^2 F}{db^2} + b \frac{dF}{db} + (1 + \mu)F = 0, \quad (\lambda RC = \mu). \]
Again, writing \( F = e^{-b/4}G \) we have for \( G \) the differential equation:

\[
\frac{d^2G}{db^2} + \left( \mu - \frac{b^2}{4} + \frac{1}{2} \right) G = 0.
\] (9)

For \( \mu = 0, 1, 2, \ldots, n \) this is the familiar differential equation of the functions of the parabolic cylinder. Let \( G_1, G_2, \ldots, G_n \) represent the normalized characteristic functions of (9), forming a complete set of orthogonal functions, belonging respectively to the characteristic values \( \mu_1, \mu_2, \ldots, \mu_n \) and satisfying the conditions (7a).

The general solution of equation (5) satisfying the boundary conditions (7a, b) can then be expressed in the form

\[
P = \sum_{1}^{\infty} A_n e^{-\mu_n \tau/RC} G_n(b) e^{-b/4}.
\]

Now since a \( \delta \)-function can always be built up from any complete set of normalized orthogonal functions according to

\[
\delta(b) = \sum_{n=1}^{\infty} G_n(b) G_n(0) e^{-b/2},
\]

it follows that the solution satisfying the boundary conditions is

\[
P_1 = \sum e^{-\mu_n \tau/RC} G_n(b) G_n(0) e^{-b/2}.
\] (10)

The probability that the voltage never passed the value \(-b_1\) during the time interval \((0, t)\) is then

\[
\int_{-b_1}^{0} P_1(b, \tau) \, db = \sum e^{-\mu_n \tau/RC} G_n(0) \int_{-b_1}^{0} G_n(b) e^{-b/2} \, db.
\] (11)

The nature of the dependence of the characteristic values \( \mu_n \) on the value of \( b_1 \) can be found by following a procedure developed by Sommerfeld \(^5\) in his studies on the Kepler problem and applied by Chandrasekhar \(^6\) to a first-passage problem \(*\) analogous to ours.

The functions of the parabolic cylinder are

\[
\Psi_n(x) = \frac{e^{-x/2} H_n(x)}{\sqrt{n! (2\pi)^{1/4}}}
\]

where the functions \( H_n \) are the Hermite polynomials, defined by \( e^{-x^2/2} H_n(x) = (-d/dx)^n e^{-x^2/2} \). If now \(-b_1\) coincides with a zero of one of the Hermite polynomials then the corresponding function \( \Psi_n \) will satisfy the boundary conditions (7a). Thus, since \( H_2 = x^2 - 1 \) has a zero at \(-1, \mu = 2\) is a characteristic value of our problem for \( b_1 = 1 \). Similarly, the higher-

\* In am indebted to Prof. Siegert for drawing my attention to this paper.
order Hermite polynomials provide such special solutions. The advantage in obtaining these special solutions is that by plotting the zero of the various Hermite polynomials \(^7\) in a \((\mu, b_1)\) diagram, we obtain at once a general indication as to how the various characteristic values are modified by the “artificial” boundary conditions at \(-b_1\) (fig. 2).

![Graph showing dependence of characteristic values on \(b_1\)](image)

Fig. 2. Dependence of the characteristic values \(\mu_n\) on the value of \(b_1\) (cf. eqs 7, 8, 9). To define the curves we have plotted the zeros of the Hermite polynomials against their order. Also we have used some of the points calculated in the text.

In equation (9) we put \(G = e^{-b_1/4} H(x)\) and we get the equation that as we have seen for \(\mu = n\) and suitable boundary conditions gives the Hermite polynomials

\[
\frac{d^2H}{db^2} - b \frac{dH}{db} + \mu H = 0.
\]

The first characteristic value of \(\mu\) depends on \(b_1\), but for \(b_1 \geq 2\) it will be near zero. Thus in the series solution

\[
H = \left\{1 - \frac{\mu}{2} b^2 - \frac{\mu(2-\mu)}{4!} b^4 - \frac{\mu(2-\mu)(4-\mu)}{6!} b^6 - \ldots\right\} + \left\{a_1 \right\}
\]

\[
+ a_1 \left\{b + \frac{1-\mu}{3!} b^3 + \frac{(3-\mu)(1-\mu)}{5!} b^5 + \ldots\right\}
\]

we can rearrange the terms so as to obtain a power series in \(\mu\) and take only
the first few terms. Then we choose \( \mu \) and \( a_1 \) in such a way that we get zeros at \(+\infty\) and \(-b_1\). The series in \( b \) which we get as a coefficient of \( \mu^n \) can easily be written in integral form. In this way one finds:

\[
H = 1 - \mu \int_0^b e^{\nu y/2} \, du \int_0^\nu e^{-\nu y/2} \, dv + a_1 \int_0^b e^{\nu y/2} \, du + ...
\]

\[
+ \mu^2 \int_0^b e^{\nu y/2} \, du \int_0^\nu e^{-\nu y/2} \, dv \int_0^u e^{\nu y/2} \, dw \int_0^v e^{-\nu y/2} \, dz - ...
\]

\[
- a_1 \mu \int_0^b e^{\nu y/2} \, du \int_0^\nu e^{-\nu y/2} \, dv \int_0^v e^{\nu y/2} \, dw + ...
\]

Therefore

\[
a_1 = \mu \int_0^\infty e^{-\nu y/2} \, dv,
\]

\[
1 - \mu \int_0^{b_1} e^{\nu y/2} \, du \int_0^\nu e^{-\nu y/2} \, dv + \mu^2 \int_0^{b_1} e^{\nu y/2} \, du \int_0^\nu e^{-\nu y/2} \, dv \int_0^u e^{\nu y/2} \, dw \int_0^v e^{-\nu y/2} \, dz + ... = 0.
\]

Partly by using tables \(^8\), and partly by using the series development we get for

\[
b_1 = 2:1 \quad 10:428 \quad \mu + 1:092 \quad \mu^2 + ... = 0; \mu = 0:10157
\]

\[
b_1 = 3:1 \quad 86:928 \quad \mu + 96:294 \quad \mu^2 + 38:712 \quad \mu^3 + ... = 0; \mu = 0:011654
\]

\[
b_1 = 4:1 \quad 2019:23 \quad \mu + 3050 \quad \mu^2 - = 0; \mu = 0:0004956
\]

\[
b_1 = 5:1 \quad 140742 \quad \mu = 0; \mu = 0:0000711.
\]

For larger values of \( b_1 \):

\[
H_{no} = aH(x); \quad a^2 \int_{-b_1}^b e^{-3y/2} H^2(x) \, dx = 1.
\]

This has been done numerically; \( a \) is a function of \( b_1 \) and

\[
a(2) = 0:6078
\]

\[
a(3) = 0:6250
\]

\[
a(4) = 0:6311
\]

\[
a(5) = 0:6316
\]

For larger values of \( b_1 \), \( a = (2\pi)^{-1/4} = 0:6316 \).

Thus for \( b_1 = 2 \), the first term in \( P_1 \) (eq. 10) is:

\[
e^{-b_1/2} e^{0:10157x/RC} 0:6078 \int_0^b e^{\nu y/2} \, du \int_0^\nu e^{-\nu y/2} \, dv + ...
\]

and in \( \int_{-b_1}^b P_1(b, \tau) \, db \) the first term is (equation 11):

\[
0:9525 e^{0:10157x/RC}.
\]
Analogously for

\[
\begin{align*}
    b_1 &= 3 & \text{the same first term is } & 0.9889 \; e^{-0.011654t/RC} \\
    b_1 &= 4 & & 0.9992 \; e^{-0.000495t/RC} \\
    b_1 &= 5 & & 0.99999 \; e^{-0.00000711t/RC} \\
    b_1 &= \sqrt{40} & & e^{-5.063.10^{-2}t/RC}.
\end{align*}
\]

From these results it is obvious that the first term in the development (10) is the most important. The second term in the development corresponds to a value of \( \mu \) near 1, therefore \( G_2(0) \) will be near zero. The third term is nearly proportional to \( e^{-2t/RC} \) and therefore dies out soon. These higher-order terms define the initial behaviour of \( P_1 \), but to calculate this we shall follow another approach, which has been suggested by Schrödinger in a paper on Brownian motion.

3. The Schrödinger integral equation

The probability that at the time \( t \) the voltage is below \( -b_1 \) (in our normalized coordinates) is the integral over \( \tau \) of the product of

(a) the probability \( p \) that it has passed the value \( -b_1 \) for the first time at a time \( \tau \) between 0 and \( t \), and

(b) the probability that the voltage is below \( -b_1 \) when it was at \( -b_1 \) a time \( t-\tau \) beforehand.

\[
\int_{-\infty}^{-b_1} P(0|b,t) db = \int_0^t d\tau P(0|\tau) \int_{-\infty}^{-b_1} P(-b_1|b,t-\tau) d\tau. \tag{12}
\]

By integration of equation (6) we get

\[
1 - \Phi \left\{ \frac{b_1}{2\sqrt{t}(1-e^{-2t/RC})} \right\} = \int_0^t d\tau P(0|-b_1,\tau) \left[ 1 - \Phi \left\{ \frac{b_1(1-e^{-(t-\tau)/RC})}{2\sqrt{t}(1-e^{-2(t-\tau)/RC})} \right\} \right]. \tag{13}
\]

Here

\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx.
\]

The form of equation (13) suggests the use of the Laplace transform, as Siegert has pointed out. Since, however, the Laplace transforms of the functions discussed here are not available explicitly, this solution is rather formal.

For small values of \( t \) we get

\[
\int_0^t d\tau P(0|-b_1,\tau) \approx 1 - \Phi \left\{ \frac{b_{1t}/RC}{2\sqrt{t}(1-e^{-2t/RC})} \right\}; \quad (b_{1t}/RC \ll 1).
\]
For an approximate solution we take $RC$ as unit of time and solve step by step. The integral equation has the general form

$$F(t) = \int_0^t p(\tau) f(t-\tau) \, d\tau,$$

where $F$ and $f$ are known functions.

We approximate

$$F(0.1) = \int_0^{0.05} p(0.05) f(\tau) \, d\tau; \quad p(0.05) = \frac{F(0.1)}{\int_0^{0.05} f(\tau) \, d\tau},$$

$$F(0.2) = \int_0^{0.15} p(0.15) f(\tau) \, d\tau + \int_0^{0.05} p(0.05) f(\tau) \, d\tau; \text{ etc.}$$

The time division scale has been taken in accordance with the character of the problem. A second approximation is given by

$$F(0.1) = \int_0^{0.05} p(0.05) f(\tau) \, d\tau + \frac{1}{12000} p'(0.05) f'(0.05); \text{ etc.,}$$

where $p'$ is estimated from the first approximation. In our application this had only a very small effect.

We calculated the following approximate solutions:

<table>
<thead>
<tr>
<th>$\tau/RC$</th>
<th>$p(0.2, \tau)$</th>
<th>$10^6 , p(0.4, \tau)$</th>
<th>$10^9 , p(0.2\sqrt{10}, \tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.0000343</td>
<td>0.00000647</td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.00643</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.03295</td>
<td>0.0465</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td>0.06248</td>
<td>1.234</td>
<td>0.0000000419</td>
</tr>
<tr>
<td>0.45</td>
<td>0.08307</td>
<td>8.092</td>
<td>0.0000481</td>
</tr>
<tr>
<td>0.55</td>
<td>0.09502</td>
<td>26.18</td>
<td>0.001047</td>
</tr>
<tr>
<td>0.65</td>
<td>0.1011</td>
<td>56.98</td>
<td>0.008287</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1036</td>
<td>97.62</td>
<td>0.03685</td>
</tr>
<tr>
<td>0.85</td>
<td>0.1041</td>
<td>143.5</td>
<td>0.1070</td>
</tr>
<tr>
<td>0.95</td>
<td>0.1036</td>
<td>190.3</td>
<td>0.2491</td>
</tr>
<tr>
<td>1.05</td>
<td>0.1020</td>
<td>235.1</td>
<td>0.4660</td>
</tr>
<tr>
<td>1.15</td>
<td>0.1003</td>
<td>276.0</td>
<td>0.7507</td>
</tr>
<tr>
<td>1.25</td>
<td>0.0984</td>
<td>312.2</td>
<td>1.081</td>
</tr>
<tr>
<td>1.35</td>
<td>0.0965</td>
<td>343.5</td>
<td>1.481</td>
</tr>
<tr>
<td>1.45</td>
<td>0.0947</td>
<td>370.2</td>
<td>1.869</td>
</tr>
<tr>
<td>1.55</td>
<td>0.0928</td>
<td>392.6</td>
<td>2.257</td>
</tr>
<tr>
<td>1.65</td>
<td>0.0911</td>
<td>411.3</td>
<td>2.624</td>
</tr>
<tr>
<td>1.75</td>
<td>0.0894</td>
<td>426.8</td>
<td>2.968</td>
</tr>
</tbody>
</table>
This initial behaviour is illustrated in figs 3, 4, 5. By integration we get the probability that the value \( b_1 \) has been passed at least once, viz.

\[
\int_0^1 p(0|2, \tau) d\tau = 0.06913; \quad \int_{15}^5 p(0|2, \tau) d\tau = 0.04923;
\]
\[ \int_0^1 p(0|4, \tau) \, d\tau = 52.45 \cdot 10^{-6}, \quad \int_1^2 p(0|4, \tau) \, d\tau = 0.0003657, \]
\[ \int_1^{2.5} p(0|4, \tau) \, d\tau = 0.0002351. \]

And if \( h(n) = \int_{n=1}^{n} p(0|2\sqrt{10}, \tau) \, d\tau, \) then \( h(1) = 4 \cdot 0.10^{-11}; \) \( h(2) = 2 \cdot 0.03 \cdot 10^{-9}; \)
\( h(3) = 4 \cdot 42 \cdot 10^{-9}; \) \( h(4) = 4 \cdot 93 \cdot 10^{-9}; \) \( h(5) = 5 \cdot 81 \cdot 10^{-9}. \)

Fig. 4. The probability \( p(0|4, \tau) \) that the voltage passes the value \(-4\) (normalized coordinates) for the first time after a time interval \((0, \tau)\) when starting at \( t = 0 \) with the value 0.

Fig. 5. The probability \( p(0|2\sqrt{10}, \tau) \) that the voltage passes the value \(-2\sqrt{10}\) (normalized coordinates) for the first time after a time interval \((0, \tau)\) when starting at \( t = 0 \) with the value 0.
The approximate value of the maximum of \( p(0|b_1, \tau) \), which as we have seen is not very sharp, we can get directly from the integral equation.

If

\[
1 - \Phi\left( \frac{b}{\sqrt{2}} \right) = \gamma, \quad \text{and} \quad \int_{0}^{\infty} \left\{ \Phi\left( \frac{b}{\sqrt{2}} \right) - \Phi\left( \frac{b}{\sqrt{2}} \left( 1 - e^{-u} \right)^{\frac{1}{2}} \right) \right\} du = \beta,
\]

then

\[ p_{\text{max}}(0|b_1, \tau) \approx \gamma/\beta. \]  

Also by solving the integral equation

\[
\gamma U(t) = \int_{0}^{t} p(0|b_1, \tau) \left\{ U(\tau) + (\gamma - 1) U(\tau - \beta) \right\} d\tau,
\]

as can easily be done by use of the Laplace transform, we see that an approximate solution for the probability that the value \( b_1 \) never has been reached during the time \( t \) is

\[
\int_{-b_1}^{\infty} P_1(b, t) db \approx e^{-\gamma \beta}. \]  

This approximation should be valid for \( t \gg \beta \). It yields

\[
\begin{array}{c|c}
 b_1 & P_{\text{max}} = \gamma/\beta \\
 2 & 0.1045 \\
 3 & 0.011690 \\
 4 & 0.00049603 \\
 5 & 0.00000716 \\
 6 & 0.0000003538 \\
 \sqrt{40} & 0.00000005077 \\
 7 & 0.000000006259.
\end{array}
\]

Recalling that in this section the time \( t \) has been measured in units of \( RC \), we see that these results agree with the more accurate ones of section 2.

4. Applications and corrections

In shot-effect problems the calculated values refer to very small voltage variations since \( b = CAE/2ae \), where \( e = 1.59.10^{-19} \) Coulomb (the charge of the electron), and \( a^2 = EC/e \) is very large. The assumption that in eq. (4) only the first term is of importance is justified.

If a condenser is charged by pulses, \( a = Eb/2!AE \), and if we allow only deviations of 1%, again the first term is sufficient. E.g. a condenser is charged by 30 pulses per minute; what is the probability that the voltage will have dropped 1% during the time \( t \), if it started at \( t = 0 \) at the equilibrium voltage? Now an \( RC \) time of 40000 seconds corresponds to the case \( b_1 = 2 \), an \( RC \) time of 90000 seconds to the case \( b_1 = 3 \), etc. If the number of
pulses is only 12 per minute and we allow for a 10% drop in voltage, the same values of $b_1$ refer to $RC$ times of 1000 respectively 2250 seconds, since $b_1^2 = 2 RC \nu (\Delta E_{\text{max}})^2/E^2$. Whether the approximation we have used so far is sufficient depends mainly on $b_1/a$, or on $\Delta E_{\text{max}} \gamma /2E$, and in the case of a 10% permitted drop in voltage, already a marked deviation appears. This is reflected mainly in a change in $\gamma$, the probability of a voltage below $-b_1$ (the limit for $t \to \infty$ of the left-hand side in eq. 12) as we have to use more terms from the series (eq. 4) for our calculation. Taking this into account the corrected values for a 10% variation in one direction can be calculated from eqs (14) and (15), where

\[
\begin{array}{c|c}
 b_1 & p_{\text{max}} = \gamma / \beta \\
\hline
 2 & 0.1123 \\
 3 & 0.01403 \\
 4 & 0.006898 \\
 5 & 0.0001187 \\
 6 & 0.000007388 \\
 \sqrt{40} & 0.000001129 \\
 \end{array}
\]

(corrected values if $\Delta E_{\text{max}}/E = 0.1$).

\[Eindhoven, \ January \ 1950\]

REFERENCES


