INSTABILITY OF THIN-WALLED CYLINDERS SUBJECTED TO INTERNAL PRESSURE *)

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Summary

It is shown that under certain conditions a thin-walled cylinder may buckle when subjected to internal pressure. The critical value of this pressure can easily be calculated by means of Euler's well-known formula. Most of the formulae for pressure-loaded cylinders given in current text-books, however, fail to predict this behaviour and should therefore be applied with caution in the case of great lengths and/or high pressures.

Résumé

Dans certaines conditions, un cylindre à minces parois soumis à des pressions intérieures peut être sujet à „flambage“. La valeur critique de cette pression peut se calculer facilement à l'aide de la formule bien connue de Euler. Toutefois, la plupart des formules, indiquées dans les manuels usuels pour le calcul des cylindres soumis à pression, ne soufflent mot de ce flambage; aussi, dans le cas de cylindres très longs et de pressions élevées ne les utilisera-t-on qu'avec certaines précautions.

Zusammenfassung

Es wird gezeigt, daß unter bestimmten Bedingungen ein dünnwandiger Zylinder knicken kann, wenn er innerem Druck ausgesetzt wird. Der kritische Wert dieses Druckes läßt sich auf Grund der bekannten Eulerschen Formel leicht errechnen. Die meisten der in den einschlägigen Lehrbüchern enthaltenen Formeln für druckunterworfone Zylinder berücksichtigen nicht die Möglichkeit dieses Verhaltens und sind deshalb in Fällen großer Länge und bzw. oder hohen Druckes mit Vorsicht anzuwenden.

1. Introduction

The problem of elastic stability of thin-walled cylinders subjected to radial and axial forces has occupied many investigators since the middle of the last century 1). A number of well-known scientists have in turn made such contributions as would lead one to consider this problem — at least in the idealized case of simple boundary conditions and perfectly straight cylinders — as being solved. Yet in current text-books, such as those of Biezeno-Ürammel 2), Flügge 3), Love 4), Pflüger 5), Prescott 6) and Timoshenko 7), formulae are given which are far from identical. It is true that always a similar relation is found from which the critical combinations of the radial and axial pressures as a function of the dimensions of the cylinder can be derived, but considered in detail the coefficients occurring in this relation show mutual differences.

These differences are of minor importance in the case where the radial

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pressure is exerted on the cylinder externally. When, however, the formulae are applied to the special case of a cylinder subjected to internal pressure, the mutual differences appear to be really of very essential importance. In this respect the formulae may be divided into two groups, viz.

(a) the group according to which the thin-walled cylinder unloaded in the axial direction would have to buckle under a certain internal pressure (Flügge, Pflüger, Prescott);

(b) the group according to which an internal pressure would never lead to instability (Biezeno-Grammel, Love, Timoshenko).

As far as the author knows, the possibility of a thin-walled cylinder buckling under an internal pressure has never before been considered in literature. Yet this case of instability is very obvious, as will be shown in section 2 in an elementary way. This means that the formulae of the last-mentioned group are inadequate to give a useful answer under all circumstances. With cylinders of great length and/or high pressure these formulae have to be applied with caution.

The more general formulae for thin-walled cylinders subjected simultaneously to radial and to axial pressures, which formulae are to be preferred on account of the foregoing remarks, will be submitted to a closer examination in section 3. Further, for one set of deviating formulae the cause of the essential differences in the result will be given in the appendix.

Though the discussion of the problem of instability of bellows subjected to internal pressure has been deferred for another paper, it is to be mentioned that it was the investigation into the cause of the — more or less unexpected — buckling of this elastic element which drew the author's attention to the similar behaviour of thin-walled cylinders.

2. The thin-walled cylinder subjected to internal radial pressure

Imagine a cylinder closed at the ends and filled with a liquid under pressure. Unless special precautions are taken the wall will then have to bear not only the radial pressure but also an axial load. Since in the case of loading to be investigated here it is desired to confine ourselves to the radial pressure, at least at one end of the cylinder a frictionless piston will have to be applied. The liquid pressure acting in the axial direction is then taken up by the piston rod and not by the cylinder wall. If a cylinder of diameter $2a$, length $l$ and wall thickness $h(h \ll a)$ is subjected to an internal pressure $q'$, the axial force amounts to $P = \pi a^2 q'$ and thus we get the situation as represented diagrammatically in fig. 1.

From the calculation given below it appears that as the pressure is gradually increased the cylinder remains straight until, upon a certain pressure being reached, it suddenly deflects. This happens in such a way that the shape shows a great similarity to that of the straight bar buckling under
an axial load. If, for instance, both ends are hinged the cylinder assumes the shape given in fig. 2, the cross-section remaining circular and the centre line bending according to the curve \( y = f(z) \).

If the cylinder filled with liquid is cut at \( z \) according to a plane normal to the centre line, at this cross-section the normal force \( N \approx P \), the transverse force \( D \approx -Pdy/dz \) and the bending moment \( M = Py \) have to be applied in order to maintain the equilibrium. The liquid, which owing to the axial discharge of the cylinder wall transmits the whole of the normal force, naturally cannot contribute towards the transverse force and the bending moment, so that these two load components have to be taken up completely by the cylinder wall.

The moment of inertia of the ring-shaped section with respect to one of its diameters amounts to \( I = \pi a^3h \). With Young's modulus of elasticity \( E \), the rigidity of the cylinder with respect to bending is equal to \( EI \). Thus, if the cylinder buckles the curvature \( d^2y/dz^2 \) of its centre line must, when ignoring the shear effect, satisfy the relation

\[
\frac{d^2y}{dz^2} = -\frac{M}{EI}.
\]

With \( M = Py \) this leads to the differential equation

\[
\frac{d^2y}{dz^2} + \frac{Py}{EI} = 0.
\]

For the boundary conditions given (\( y = 0 \) at \( z = 0 \) and \( z = l \)) a deflection appears to be possible only if \( P \) satisfies the relation

\[
P = \pi^2EI/l^2,
\]

(1)
or, on account of the relations \( P = \pi a^2 q' \) and \( I = \pi a^3 h \), if the internal pressure \( q' \) is equal to
\[
q' = \pi a^2 \frac{Eah}{l^2}.
\]
(2)

Since the method of calculation given here agrees entirely with that followed in deriving the condition for instability of the thrust-loaded straight rod, it is not surprising that eq. (1) is identical with Euler’s well-known formula. This means, therefore, that instability likewise occurs when the critical force \( P \) given in (1) is not transferred into a radial pressure by means of the piston and the liquid, but is directly applied to the wall as an axial load. This will be reverted to in the next section.

3. The thin-walled cylinder subjected to radial and axial pressures

From the theory of shells it is known that the thin-walled cylinder subjected to radial and axial pressures may not under all circumstances be regarded as a profiled rod. The wall may be divided both along the circumference and in the longitudinal direction into a number of fields where the wall is bent alternately inward and outward.

If, in the middle of the wall given by the coordinates \( a, \varphi \) and \( z \), we denote the additional components of the displacement in the radial, tangential and axial directions respectively by \( u_0, v_0 \) and \( w_0 \), the deformation mentioned can be described by the relations (cf. Biezeno and Grammel 2), p. 603
\[
\begin{align*}
u_0 &= U \cos p\varphi \sin (\lambda z/a), \\v_0 &= V \sin p\varphi \sin (\lambda z/a), \\
w_0 &= W \cos p\varphi \cos (\lambda z/a),
\end{align*}
\]
(3)
where \( U, V \) and \( W \) are constants, and \( p \) and \( n \) each represent a positive integer. In the special case where \( p = 1 \) and \( n = 1 \) the cross-section of the cylinder remains circular and the centre line is bent according to half a sine. Thus this form of deflection is the same as that discussed in the previous section.

In the more general case where the cylinder is loaded not only with a radial pressure \( q = -q' \) but also with the axial compressive force \( Q \) per unit circumference, Flügge’s calculation 3), for instance, leads, with Poisson’s ratio \( \nu \) and the notations
\[
q_1 = \frac{(1 - \nu^2)qa}{Eh}, \quad q_2 = \frac{(1 - \nu^2)Q}{Eh}, \quad k = \frac{h^2}{12 a^2},
\]
(4)

\( q_1 \) and \( q_2 \) to the condition of instability
\[
c_1 + c_2 k = c_3 q_1 + c_4 q_2.
\]
(5)

The coefficients \( c_i (i = 1, 2, 3, 4) \) occurring in this equation are determined,
in a somewhat more complete form than that given originally by Flügge, by

\[
\begin{align*}
\mathcal{C}_1 &= (1-v^2)\lambda^4, \\
\mathcal{C}_2 &= (\lambda^2+p^2)^4 - 2\lambda^2p^2 + (4-v)\lambda^2p^6 + \\
&\quad + (4-3v^2)\lambda^4 + 2(2-v)\lambda^2p^2 + p^2, \\
\mathcal{C}_3 &= (\lambda^2+p^2)^2(p^2-1) - \lambda^2p^2 + 2\lambda^2, \\
\mathcal{C}_4 &= (\lambda^2+p^2)^2\lambda^2 + 2(1+v)\lambda^4 + \lambda^2p^2.
\end{align*}
\]

(6)

To apply this result as a check upon the elementary case given in the previous section we need only introduce the substitutions \( p = 1 \), \( n = 1 \), \( \lambda \ll 1 \) and \( k \ll 1 \). It is then found that to a first approximation instability occurs if

\[
-q_1 + 2q_2 \approx (1-v^2)\lambda^2,
\]

or — owing to \( q = -q' \), \( \lambda = \pi a/l \) and eq. (4) — if

\[
q' + 2Q/a \approx \pi^2 E\pi a/l^2.
\]

(8)

For the cylinder wall unloaded in the axial direction \( Q = 0 \) applies, so that the result in this case is identical with the result (2) deduced in the elementary way. On the other hand when the cylinder wall is axially loaded \( \pi a^2(q' + 2Q/a) \) represents the total external force applied axially on the ends. Thus, it appears to be this total force which, substituted in Euler’s well-known formula, gives the condition of instability (8), so that with respect to this instability it is immaterial whether the force is exerted directly on the wall or transmitted entirely or partly by means of a piston to the liquid contained in the cylinder.

If the cylinder is closed at its ends, a case which often occurs, the wall has to bear per unit of circumference an axial tensile force \( -Q = \frac{1}{2} q' a \). From eq. (8) it follows that this tensile force is just sufficient to cancel the buckling effect of the radial pressure. Probably this will be one of the reasons why the possibility of a cylinder buckling when subjected to internal pressure has not become common experience.

Flügge explains the results of his calculation by means of diagrams which take into account the presence of internal pressures \( q' < 0 \). But then he is considering only the effect of these pressures upon the axial compressive force, without mentioning that an internal radial pressure may in itself cause the buckling of a thin-walled cylinder unloaded in the axial direction. It is therefore to be concluded that this possibility also escaped his notice.

A comparison of Flügge’s results with, for instance, those of Biezeno and Grammel 2) shows that the condition of buckling given by the last-mentioned authors can also be written in the form of eq. (5) with the
coefficients $c_1$, $c_2$ and $c_4$ according to (6) but with the coefficient $c_3$ equal to

$$ c_3 = (\lambda^2 + p^2) (p^2 - 1) + \nu \lambda^4. $$

The term $\lambda^2 p^2$ required for the result (7) has disappeared from the equation for $c_3$ and so the formula of Biezeno and Grammel does not lead to the desired result. The reason for this discrepancy is briefly explained in the appendix.

Whilst Plüger 5) and Prescott 6) confirm the result of Flügge's calculation, that the thin-walled cylinder unloaded in the axial direction will buckle under a certain internal pressure, Love 4) and Timoshenko 7) give as a result, in accordance with the Biezeno-Grannmel calculation, that in no case will an internal pressure lead to instability. From the elementary calculation given in section 2 it is clear that only the group of authors first mentioned have the right view of the problem.

Here it is to be emphasized that the differences described find expression only in the special case of internal pressure, because only then is the solution $p = 1$ a possible one. In the cases where the radial pressure is exerted on the cylinder externally we always have $p \neq 1$ and the mutual differences are of minor importance.

4. Summary

The problem of the thin-walled cylinder subjected to radial and axial pressures is dealt with in many text-books for applied mechanics.

For the idealized case of simple boundary conditions and perfectly straight cylinders it is possible to give a solution which in general is fairly reasonable. There appear to be, however, mutual differences in the formulae given, and, although these differences are in many cases of minor importance, in the case of a cylinder loaded by internal pressure, as dealt with here, they lead to essentially different results.

By means of an elementary calculation where the thin-walled cylinder is regarded as a profiled straight rod, it can be shown that the cylinder unloaded in the axial direction will buckle under a certain internal pressure. This phenomenon is predicted by only a limited number of formulae [e.g. Flügge 3), Pflüger 5) and Prescott 6)], which are therefore to be preferred to the other group of formulae [e.g. Biezeno and Grammel 2), Love 4) and Timoshenko 7)] according to which a cylinder subjected to internal pressure would never be able to become unstable.

It appears that the internal pressure necessary for the buckling of a cylinder unloaded in the axial direction by means of a piston (fig. 1) can be calculated by putting the force exerted on the piston equal to the axial buckling force calculated from Euler's formula. When investigating the danger of buckling we have to deal with the total axial force, no matter
whether this force is transmitted directly to the cylinder wall as an axial load or, entirely or partly, by means of a piston and liquid as a radial load.

In consequence of the great rigidity of the cylinder with respect to bending, the kind of instability pointed out here will occur only in the case of high pressures and/or great lengths of the cylinder. Furthermore special precautions have to be taken to unload the cylinder wall in the axial direction entirely or partly, for if the cylinder is closed at its ends the axial tensile force in the wall is just sufficient to cancel the buckling effect resulting from the radial pressure.

Appendix

The difference between the results of Flügge 3 and those of Biezeno and Grammel 2 occurs because the last-mentioned authors have taken no account of the axial component $-q' \partial u/\partial z$ of the hydrostatic pressure when calculating the additional load in the deflected state of the wall. When their second eq. (VII, 30, 6) with $q = -q'$ is replaced by

$$Z = -\frac{q_0^2 w_0}{a \partial \varphi^2} + q \frac{\partial u_0}{\partial z},$$

the coefficient of $U$ in their third eq. (VII, 30, 10) increases with the term $\lambda q_1$. Supplemented in this way their eqs (VII, 30, 10) conform to the corresponding equations of Flügge.

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REFERENCES

1) For a brief historical survey see, e.g., Sanden and Folke, Über Stabilitätsprobleme dünner kreiszylindrischer Schalen, Ing. Archiv 3, 24, 1932.


8) After the completion of this paper I learned from Dr A. Klinkenberg of the Bataafse Petroleum Maatschappij (The Hague) that the buckling of drill pipes had been a point of discussion at the Spring meeting of the Southwestern District, Division of Production, American Petroleum Institute (March 1951). In commenting upon Klinkenberg’s paper Mr H. B. Woods gave a formula for the buckling of drill pipes. According to this formula buckling of a thin-walled cylinder may occur when the axial tensile force per unit of circumference is smaller than half the difference between the internal and external hydrostatic pressures multiplied by the radius of the cylinder. This limit agrees with relation (8) for infinitely long cylinders.