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A GENERAL NETWORK THEOREM, WITH APPLICATIONS

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Summary

It is proved that in a network configuration, for branch currents i satisfying the node equations and branch voltages v satisfying the mesh equations, Σiv summed over all branches is zero. By this theorem it is possible to prove the energy theorem and the reciprocity relation of networks, and to show that if given, arbitrarily varying voltages are applied on a $2n$-pole at rest the difference between the electric and the magnetic energy will at any instant depend only on the admittance matrix of the $2n$-pole and not on the particular network used for realizing it.

Résumé

Il a été démontré que dans une configuration de réseau pour courants de branche i répondant aux équations nodales et pour tensions de branche v répondant aux équations de maille, Σiv sommé sur tous les branches est égal à zéro. Ce théorème permet de prouver le théorème de l'énergie et la relation de réciprocité des réseaux et de démontrer que si des tensions données, variant arbitrairement, sont appliquées à un $2n$-pôle au repos, la différence entre les énergies électrique et magnétique dépendra à tout instant seulement de la matrice d'admittance du $2n$-pôle et non pas du réseau particulier, utilisé pour sa réalisation.

Zusammenfassung

Es wird gezeigt, daß in einer Netzwerkkonfiguration bezüglich Zweigströme i, die den Knotengleichungen genügen, und Zweigspannungen v, die den Maschengleichungen genügen, Σiv, summiert über alle Zweige, gleich Null ist. Dieses Theorem ermöglicht es, das Energietheorem und die Reziprozitätsbeziehung von Netzwerken nachzuweisen, sowie zu zeigen, daß bei Anwendung gegebener, willkürlich veränderbarer Spannungen auf einen $2n$-Pol in Ruhe, die Differenz zwischen der elektrischen und der magnetischen Energie in jedem Zeitpunkt ausschließlich von der Admittanzmatrix des $2n$-Poles abhängt und nicht von dem besonderen Netzwerk, das zu ihrer Verwirklichung benutzt wird.

1. Introduction

Many properties of electrical networks can be derived from their impedance functions alone, without a detailed knowledge of their circuit diagrams. Certain properties, however, cannot be established in this way; for example, in order to prove the energy theorem and the reciprocity relation of networks and to investigate the distribution of the supplied energy in electric, magnetic, and dissipated energy, the inner structure of the network has to be taken into account. In this paper a general theorem is given which can be used in such cases. Considerations leading to the theorem are inherently present in various network investigations; the theorem itself was, so far as the author is aware, never explicitly stated.

2. The general theorem

We prove the following

Theorem:

In a network configuration, imagine branch currents i such that for every node $\Sigma i = 0$, imagine branch voltages v such that for every mesh $\Sigma v = 0$, and for every branch let the positive direction of the current be from the + to the — denoting the positive polarity of the voltage (fig. 1). Then $\Sigma iv = 0$, where the summation is over all branches.

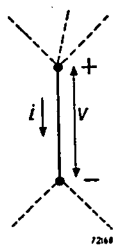

Fig. 1. Positive senses of current and voltage of a branch.

Since for every mesh $\Sigma v = 0$, there exist node potentials V such that the voltage on each branch is equal to the difference between the potentials of its end points. We denote the potentials of the nodes k and l by V_k and V_l, and the voltage on the branch connecting k and l by v_{kl}, where $v_{kl} = V_k - V_l$. We denote the current in the branch flowing from k to l by i_{kl}. Then

$$i_{kl}v_{kl} = i_{kl}(V_k - V_l) = i_{kl}V_k + i_{lk}V_l \, .$$

After applying this to all terms of Σiv, we collect the terms containing V_k, $\Sigma i_{kl}V_k = V_k\Sigma i_{kl}$, where the summation is over all nodes that are connected to k by a branch. Since the i_{kl} now considered are the currents flowing away from k, their sum is zero and thus the sum of the considered terms of Σiv is zero. The same holds for the sum of the terms containing any other node potential, so that $\Sigma iv = 0$.

The theorem is the network equivalent of the well-known theorem that the volume integral of the scalar product of a solenoidal vector (comparable with i) and an irrotational vector (comparable with v) is zero. The theorem holds for all types of network, linear and nonlinear, constant and variable, passive and active. The i's and v's in the theorem may be complex quantities satisfying the node equations and the mesh equations, respectively. The theorem can be applied to networks provided with terminal pairs by considering the terminal pairs as constituting branches of the network. If we take the positive senses of the current and the voltage of a terminal pair as indicated in fig. 2, we can write the theorem as $\Sigma i_t v_t = \Sigma i_b v_b$, where the first summation is over the terminal pairs and the second over the internal branches.

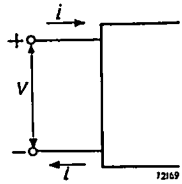

Fig. 2. Positive senses of current and voltage of a terminal pair.

3. The energy theorem of networks

If i and v denote branch currents and voltages simultaneously present in a network, $\Sigma iv = 0$ means that at any instant the total power consumption is zero. This constitutes the energy theorem of networks. For networks with terminal pairs it implies that the power absorbed by the branches is equal to the power delivered to the network through the terminal pairs.

If I and V are complex quantities representing sinusoidal branch currents and voltages of the same frequency in a constant, linear network we may write

$$\Sigma I_t V_t^* = \Sigma I_b V_b^* , \tag{1}$$

where the asterisk denotes the conjugate complex quantity, and the first summation is again over the terminal pairs and the second over the internal branches. The left-hand side of (1) equals the active power $+j$ reactive power delivered to the network. The right-hand side of (1) may be written as

$$\sum_k R_k I_k I_k^* + j\omega \sum_p C_p V_p V_p^* - j\omega \sum_{m,n} L_{mn} I_m I_n^* = W_{\text{av}} + 2j\omega (U_{\text{av}} - T_{\text{av}}) ,$$

where the R_k are the resistances, the C_p are the capacitances, and the L_{mn} are the self and mutual inductances of the network; W_{av}, U_{av}, and T_{av} are the average dissipated power, the average electric energy, and the average magnetic energy, respectively. Thus we find

$$\text{active power} = W_{\text{av}}, \quad \text{reactive power} = 2\omega\,(U_{\text{av}} - T_{\text{av}}) .$$

These results are related to results obtained by Bode [1]).

4. The reciprocity relation

Let us consider two states of a constant, linear four-pole, the first state being denoted by unprimed quantities, the second by primed quantities. We can then write

$$I_1 V_1' + I_2 V_2' = \Sigma I_b V_b' ,$$
$$I_1' V_1 + I_2' V_2 = \Sigma I_b' V_b ,$$

the subscripts 1 and 2 denoting the terminal pairs, the subscript b the internal branches. Considering a branch with impedance Z, for that branch we have

$$I_b V_b' = I_b Z I_b' = I_b' Z I_b = I_b' V_b .$$

Considering a set of n coupled coils for which

$$V_{bi} = \sum_{k=1}^{n} Z_{ik} I_{bk}, \qquad (i = 1, \ldots, n; \, Z_{ik} = Z_{ki}),$$

we have

$$\sum_{i=1}^{n} I_{bi} V_{bi}' = \sum_{i=1}^{n} I_{bi} \sum_{k=1}^{n} Z_{ik} I_{bk}' = \sum_{k=1}^{n} I_{bk}' \sum_{i=1}^{n} Z_{ki} I_{bi} = \sum_{k=1}^{n} I_{bk}' V_{bk} .$$

Considering an ideal transformer for which

$$I_{b1} = -T I_{b2},$$
$$V_{b2} = T V_{b1},$$

we have

$$I_{b1} V_{b1}' + I_{b2} V_{b2}' = -T I_{b2} V_{b1}' + I_{b2} T V_{b1}' = 0 ,$$

and also $$I_{b1}' V_{b1} + I_{b2}' V_{b2} = 0 .$$

Thus we see that for a four-pole composed of the elements under consideration

$$\Sigma I_b V_b' = \Sigma I_b' V_b ,$$

and hence

$$I_1 V_1' + I_2 V_2' = I_1' V_1 + I_2' V_2 . \qquad (2)$$

This is the reciprocity relation of four-poles in its general form. The proof shows that for this relation to hold, the four-pole need not be passive. It may contain negative resistances.

There are four important special cases of (2).
With $I_2 = 0$ and $I_1' = 0$, we get $V_2/I_1 = V_1'/I_2'$.
With $V_2 = 0$ and $V_1' = 0$, we get $I_2/V_1 = I_1'/V_2'$.
With $I_2 = 0$ and $V_1' = 0$, we get $V_2/V_1 = -I_1'/I_2'$.
With $V_2 = 0$ and $I_1' = 0$, we get $I_2/I_1 = -V_1'/V_2'$.

5. The distribution of energy

Let us consider a linear, constant, passive $2n$-pole the admittance matrix of which is given as a function of the frequency parameter λ, and let it contain no energy: no voltages on the capacitors, no currents through the coils. At $t = 0$ we apply arbitrarily varying voltages to the $2n$-pole, which causes a certain power to be consumed by it. This power is partly dissipated in the resistors, partly stored as electric energy in the capacitors and as magnetic energy in the coils. If we know the $2n$-pole network we can at any instant calculate this distribution of energy. We know that a given admittance matrix can be realized by different networks. We now ask whether anything can be said about the distribution of energy that depends only on the admittance matrix and not on the particular network used for realizing it *).

To answer the question, imagine a $2n$-pole network composed of resistances, capacitances, inductances, and ideal transformers **). At $t = 0$ constant voltages $v_1, \ldots, v_n$ are applied to the $2n$-pole. We suppose the admittance matrix to have no pole at $\lambda = \infty$ (as would be the case if the network contained capacitances in parallel to its terminal pairs). Then at $t = 0$ the $2n$-pole currents $i_1, \ldots, i_n$ remain finite.

As a consequence, immediately after applying the voltages the $2n$-pole will contain no energy, i.e. the currents through the inductances and the voltages on the capacitances will still be zero. We further suppose the admittance matrix to have no pole at $\lambda = 0$ (as would be the case if the network contained inductances in parallel to its terminal pairs). Then at $t = \infty$ the $2n$-pole currents remain finite. Admittance matrices with poles at $\lambda = \infty$ or $\lambda = 0$ can be considered as limiting cases of those without such poles, and therefore these will not be investigated separately.

Let us begin by considering a $2n$-pole the admittance matrix of which has only two poles, λ_a and λ_b. The poles may be real or conjugate complex. We suppose them to be unequal (equal poles can be considered as a limiting case of unequal ones). The poles determine the free oscillations of the $2n$-pole with short-circuited terminal pairs. For positive t every $2n$-pole current is the sum of a term with $e^{\lambda_a t}$, a term with $e^{\lambda_b t}$, and a constant term. As the $2n$-pole voltages are constant, we can write the power delivered to the $2n$-poles as

$$i_1 v_1 + \ldots + i_n v_n = G_a e^{\lambda_a t} + G_b e^{\lambda_b t} + G_0. \qquad (3)$$

*) It was this question, for a two-pole composed of resistors and capacitors only, that induced me to undertake the investigations reported in this paper. This question was put to me by Dr A. J. Staverman and Dr F. Schwarzl of "Kunststoffeninstituut T.N.O." Delft, Netherlands, who were led to it by their study of visco-elastic matter, the electric equivalent of which is a system composed of resistors and capacitors.

**) Networks containing gyrators [2]) have to be excluded from the following considerations. Since a gyrator of which one terminal pair is connected to a capacitance behaves as an inductance, in such networks the distinction between electric and magnetic energy loses its interest.

If the admittance matrix and v_1, ..., v_n are known, G_a, G_b, G_0 can be calculated in several ways.

The current and the voltage of a resistance R_k we write as

$$i_{Rk} = A_{ak}e^{\lambda_a t} + A_{bk}e^{\lambda_b t} + A_{0k}, \quad v_{Rk} = R_k(A_{ak}e^{\lambda_a t} + A_{bk}e^{\lambda_b t} + A_{0k}).$$

The current and the voltage of an inductance L_m we write as

$$i_{Lm} = B_{am}(e^{\lambda_a t} - 1) + B_{bm}(e^{\lambda_b t} - 1), \quad v_{Lm} = L_m(B_{am}\lambda_a e^{\lambda_a t} + B_{bm}\lambda_b e^{\lambda_b t}),$$

since $i_{Lm} = 0$ at $t = 0$.

The voltage and the current of a capacitance C_p we write as

$$v_{Cp} = D_{ap}(e^{\lambda_a t} - 1) + D_{bp}(e^{\lambda_b t} - 1), \quad i_{Cp} = C_p(D_{ap}\lambda_a e^{\lambda_a t} + D_{bp}\lambda_b e^{\lambda_b t}),$$

since $v_{Cp} = 0$ at $t = 0$.

We now apply our general theorem and take the currents at $t = t'$ and the voltages at $t = t''$. In section 4 it has been seen that for the ideal transformers $\Sigma iv = 0$, and so we can disregard them here. Thus we can write

$$\begin{aligned} G_a e^{\lambda_a t'} + G_b e^{\lambda_b t'} + G_0 = \\ = \sum_k R_k(A_{ak}e^{\lambda_a t'} + A_{bk}e^{\lambda_b t'} + A_{0k})(A_{ak}e^{\lambda_a t''} + A_{bk}e^{\lambda_b t''} + A_{0k}) + \\ + \sum_m L_m\{B_{am}(e^{\lambda_a t'} - 1) + B_{bm}(e^{\lambda_b t'} - 1)\}(B_{am}\lambda_a e^{\lambda_a t''} + B_{bm}\lambda_b e^{\lambda_b t''}) + \\ + \sum_p C_p(D_{ap}\lambda_a e^{\lambda_a t'} + D_{bp}\lambda_b e^{\lambda_b t'})\{D_{ap}(e^{\lambda_a t''} - 1) + D_{bp}(e^{\lambda_b t''} - 1)\}. \end{aligned}$$

Since this must hold for all positive values of t' and t'', we find

$$\begin{aligned} G_a &= \sum_k R_k A_{ak} A_{0k} - \lambda_a \sum_p C_p D_{ap}^2 - \lambda_a \sum_p C_p D_{ap} D_{bp}, \\ G_b &= \sum_k R_k A_{bk} A_{0k} - \lambda_b \sum_p C_p D_{bp}^2 - \lambda_b \sum_p C_p D_{ap} D_{bp}, \\ 0 &= \sum_k R_k A_{ak} A_{0k} - \lambda_a \sum_m L_m B_{am}^2 - \lambda_a \sum_m L_m B_{am} B_{bm}, \\ 0 &= \sum_k R_k A_{bk} A_{0k} - \lambda_b \sum_m L_m B_{bm}^2 - \lambda_b \sum_m L_m B_{am} B_{bm}, \\ 0 &= \sum_k R_k A_{ak} A_{bk} + \lambda_a \sum_m L_m B_{am} B_{bm} + \lambda_b \sum_p C_p D_{ap} D_{bp}, \\ 0 &= \sum_k R_k A_{ak} A_{bk} + \lambda_b \sum_m L_m B_{am} B_{bm} + \lambda_a \sum_p C_p D_{ap} D_{bp}, \end{aligned}$$

and three other relations which we do not need. From the last two equations it follows that

$$\sum_m L_m B_{am} B_{bm} = \sum_p C_p D_{ap} D_{bp}. \tag{4}$$

From the first four equations we then deduce

$$\lambda_a(\sum_m L_m B_{am}^2 - \sum_p C_p D_{ap}^2) = G_a, \quad \lambda_b(\sum_m L_m B_{bm}^2 - \sum_p C_p D_{bp}^2) = G_b. \tag{5}$$

The magnetic energy T is given by

$$\begin{aligned} T = \sum_m \tfrac{1}{2} L_m i_{Lm}^2 = \tfrac{1}{2} \sum_m L_m B_{am}^2 (e^{\lambda_a t} - 1)^2 + \tfrac{1}{2} \sum_m L_m B_{bm}^2 (e^{\lambda_b t} - 1)^2 + \\ + \sum_m L_m B_{am} B_{bm} (e^{\lambda_a t} - 1)(e^{\lambda_b t} - 1). \end{aligned}$$

The electric energy U is given by

$$U = \sum_p \tfrac{1}{2} C_p v_{Cp}^2 = \tfrac{1}{2} \sum_p C_p D_{ap}^2 (e^{\lambda_a t} - 1)^2 + \tfrac{1}{2} \sum_p C_p D_{bp}^2 (e^{\lambda_b t} - 1)^2 + \\ + \sum_p C_p D_{ap} D_{bp} (e^{\lambda_a t} - 1)(e^{\lambda_b t} - 1) .$$

By subtraction, using (4) and (5), we arrive at

$$T - U = \frac{G_a}{2\lambda_a} (e^{\lambda_a t} - 1)^2 + \frac{G_b}{2\lambda_b} (e^{\lambda_b t} - 1)^2. \qquad (6)$$

The quantities G_a, G_b, λ_a, λ_b occurring in the right-hand side of (6) depend, as seen above, only on $v_1, \ldots, v_n$ and the admittance matrix, but not on the particular network used to realize the latter, and so this also holds for $T—U$.

This result, derived for a $2n$-pole at rest on which at $t = 0$ constant voltages are applied and of which the admittance matrix has only two poles, remains valid in more general cases. To investigate $2n$-poles the admittance matrices of which have more than two poles, let us consider the expressions derived for the energy more closely. For an admittance matrix with only one pole, λ_a or λ_b, T only contains terms with $\sum_m L_m B_{am}^2$ or $\sum_m L_m B_{bm}^2$ respectively. If λ_a and λ_b are both poles, T also contains terms with $\sum_m L_m B_{am} B_{bm}$. Similar remarks apply to U. Because of (4), in $T—U$ these latter terms cancel, and thus (6) consists of the sum of a term arising from λ_a and a term arising from λ_b *). Thus, if the admittance matrix has more than two poles, the corresponding expression for $T—U$ will be equal to the sum of the terms arising from each pole separately, and thus (6) can immediately be extended to any number of poles. It can also be extended to variable voltages, as will be shown in section 7. Thus we have the

Theorem:

If on a linear, constant, passive, gyratorless 2n-pole at rest given, arbitrarily varying voltages are applied, the difference between the magnetic and the electric energy will at any instant depend only on the admittance matrix of the 2n-pole and not on the particular network used for realizing it.

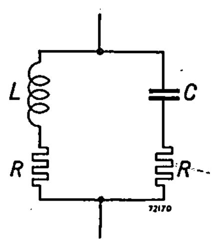

Fig. 3. Two-pole network.

*) This property is related to the conjugate property derived by Heaviside [3]).

Since for the energy A delivered to the $2n$-pole we can write $A = W+T+U$, where W is the dissipated energy, the above theorem not only holds for $T-U$ but also for $W+2T$ and for $W+2U$. If one of the energies is zero, as in resistanceless, inductanceless, or capacitanceless $2n$-poles, the theorem holds for any of the two remaining energies.

That in the general case W, T, and U are not separately determinable from the admittance matrix alone, is shown by the two-pole network of fig. 3, where $L/C = R^2$. As is well known, the impedance of the two-pole equals R. Thus, when starting from rest, T and U of the two-pole network are not determinable from the impedance function; only $T-U$ is, and here it is zero.

6. A theorem of Heaviside

From (6) we can draw an interesting conclusion. If we disregard admittance matrices that have poles at imaginary values of λ, the real parts of λ_a and λ_b will both be negative. Therefore for large t we get

$$T - U = \frac{G_a}{2\lambda_a} + \frac{G_b}{2\lambda_b}. \tag{7}$$

The energy A delivered to the $2n$-pole is determined according to (3) by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = G_a\, e^{\lambda_a t} + G_b e^{\lambda_b t} + G_0 .$$

Integrating with respect to t between the limits 0 and t, we find

$$A = \frac{G_a}{\lambda_a}\,(e^{\lambda_a t} - 1) + \frac{G_b}{\lambda_b}\,(e^{\lambda_b t} - 1) + G_0 t .$$

For large t we get

$$A = G_0 t - \frac{G_a}{\lambda_a} - \frac{G_b}{\lambda_b}. \tag{8}$$

The quantity G_0 is the power delivered to the $2n$-pole at large t, which power is dissipated. Therefore $G_0 t$ is the amount of energy that would have been dissipated up to the time t if from $t = 0$ the dissipation rate would have had the value G_0, which energy we denote by W'. From (7) and (8) it then follows that

$$A - W' = 2(U - T) . \tag{9}$$

This relation remains valid if the admittance matrix has more than two poles. It expresses a theorem due to Heaviside and proved by Lorentz [2]), which may be stated as follows:

Theorem:

If on a linear, constant, passive, gyratorless 2n-pole at rest suddenly constant voltages are applied, then, when the final state has been reached, the total energy delivered to the 2n-pole exceeds the energy representing the loss by dissipation at the final rate, supposed to start at once, by twice the excess of the electric over the magnetic energy.

7. Variable voltages

To investigate the distribution of energy when variable voltages are applied to a $2n$-pole at rest, we first consider a four-pole the admittance matrix of which has two poles, on whose first terminal pair a constant voltage v_1 is applied at $t = t_1$, and on whose second terminal pair a constant voltage v_2 is applied at $t = t_2$. For $t > t_1$ and $t > t_2$ the currents and the voltages are the superposition of those due to v_1 and those due to v_2. The currents to the terminal pairs can be written as

$$\begin{aligned} i_1 = \{G_{a11}\, e^{\lambda_a(t-t_1)} + G_{b11}\, e^{\lambda_b(t-t_1)} + G_{011}\}\, v_1 + &\\ + \{G_{a12}\, e^{\lambda_a(t-t_2)} + G_{b12}\, e^{\lambda_b(t-t_2)} + G_{012}\}\, v_2\,, &\\ i_2 = \{G_{a12}\, e^{\lambda_a(t-t_1)} + G_{b12}\, e^{\lambda_b(t-t_1)} + G_{012}\}\, v_1 + &\\ + \{G_{a22}\, e^{\lambda_a(t-t_2)} + G_{b22}\, e^{\lambda_b(t-t_2)} + G_{022}\}\, v_2\,. & \end{aligned}$$

If the admittance matrix is known, the G's can be calculated. They do not depend on v_1, v_2, t_1, t_2. The power delivered to the four-pole amounts to

$$\begin{aligned} i_1 v_1 + i_2 v_2 = {} & G_{a11}\, v_1^2\, e^{\lambda_a(t-t_1)} + G_{a12}\, v_1 v_2\, \{e^{\lambda_a(t-t_1)} + e^{\lambda_a(t-t_2)}\} + G_{a22}\, v_2^2\, e^{\lambda_a(t-t_2)} + \\ & + G_{b11}\, v_1^2\, e^{\lambda_b(t-t_1)} + G_{b12}\, v_1 v_2\, \{e^{\lambda_b(t-t_1)} + e^{\lambda_b(t-t_2)}\} + G_{b22}\, v_2^2\, e^{\lambda_a(t-t_2)} + \\ & + G_{011}\, v_1^2 + 2\, G_{012}\, v_1 v_2 + G_{022}\, v_2^2\,. \end{aligned}$$

For the current through the resistance R_k we take the same expression as in section 5.

The current through the inductance L_m we now write as

$$\begin{aligned} i_{Lm} = {} & B_{a1m} v_1\, \{e^{\lambda_a(t-t_1)} - 1\} + B_{a2m}\, v_2\, \{e^{\lambda_a(t-t_2)} - 1\}, \\ & + B_{b1m} v_1\, \{e^{\lambda_b(t-t_1)} - 1\} + B_{b2m}\, v_2\, \{e^{\lambda_b(t-t_2)} - 1\}, \end{aligned} \tag{10}$$

since $i_{Lm} = 0$ at $t = t_1$ if $v_2 = 0$, and at $t = t_2$ if $v_1 = 0$.
The voltage on the capacitance C_p we write accordingly as

$$\begin{aligned} v_{Cp} = {} & D_{a1p}\, v_1\, \{e^{\lambda_a(t-t_1)} - 1\} + D_{a2p}\, v_2\, \{e^{\lambda_a(t-t_2)} - 1\} + \\ & + D_{b1p}\, v_1\, \{e^{\lambda_b(t-t_1)} - 1\} + D_{b2p}\, v_2\, \{e^{\lambda_b(t-t_2)} - 1\}. \end{aligned}$$

As in section 5, we apply our general theorem with the currents at $t = t'$ and the voltages at $t = t''$. Instead of (4) we now arrive at

$$\sum_m L_m \,(B_{a1m}\, v_1 e^{-\lambda_a t_1} + B_{a2m}\, v_2 e^{-\lambda_a t_2})\,(B_{b1m} v_1 e^{-\lambda_b t_1} + B_{b2m} v_2 e^{-\lambda_b t_2}) =$$
$$= \sum_p C_p \,(D_{a1p}\, v_1 e^{-\lambda_a t_1} + D_{a2p}\, v_2 e^{-\lambda_a t_2})\,(D_{b1p} v_1 e^{-\lambda_b t_1} + D_{b2p} v_2 e^{-\lambda_b t_2}) .$$

Since this must hold for all values of t_1 and t_2, we find

$$\left.\begin{array}{l} \sum_m L_m B_{a1m} B_{b1m} = \sum_p C_p D_{a1p} D_{b1p},\ \sum_m L_m B_{a2m} B_{b2m} = \sum_p C_p D_{a2p} D_{b2p}, \\ \sum_m L_m B_{a1m} B_{b2m} = \sum_p C_p D_{a1p} D_{b2p},\ \sum_m L_m B_{a2m} B_{b1m} = \sum_p C_p D_{a2p} D_{b1p} . \end{array}\right\} \quad (11)$$

By means of these relations, instead of (5) we now arrive at

$$\lambda_a \Big\} \sum_m L_m (B_{a1m} v_1 e^{-\lambda_a t_1} + B_{a2m} v_2 e^{-\lambda_a t_2})\, (B_{a1m} v_1 + B_{a2m} v_2) -$$
$$- \sum_p C_p (D_{a1p} v_1 e^{-\lambda_a t_1} + D_{a2p} v_2 e^{-\lambda_a t_2})\, (D_{a1p} v_1 + D_{a2p} v_2) \Big\{ =$$
$$= G_{a11}\, v_1^2 e^{-\lambda_a t_1} + G_{a12} v_1 v_2 (e^{-\lambda_a t_1} + e^{-\lambda_a t_2}) + G_{a22} v_2^2 e^{-\lambda_a t_2},$$

and a similar equation derived therefrom by changing the index a into b. Since this must hold for all values of v_1 and v_2, we find

$$\left.\begin{array}{l} \lambda_a (\sum_m L_m B_{a1m}^2 - \sum_p C_p D_{a1p}^2) = G_{a11}, \\ \lambda_a (\sum_m L_m B_{a1m} B_{a2m} - \sum_p C_p D_{a1p} D_{a2p}) = G_{a12}, \\ \lambda_a (\sum_m L_m B_{a2m}^2 - \sum_p C_p D_{a2p}^2) = G_{a22}, \end{array}\right\} \quad (12)$$

and three similar relations by changing the index a into b. The magnetic energy and the electric energy are given by

$$T = \tfrac{1}{2} \sum_m L_m [B_{a1m} v_1 \big\} e^{\lambda_a (t-t_1)} - 1 \big\{ + B_{a2m} v_2 \big\} e^{\lambda_a (t-t_2)} - 1 \big\{ +$$
$$+ B_{b1m} v_1 \big\} e^{\lambda_b (t-t_1)} - 1 \big\{ + B_{b2m} v_2 \big\} e^{\lambda_b (t-t_2)} - 1 \big\{]^2 ;$$
$$U = \tfrac{1}{2} \sum_p C_p [D_{a1p} v_1 \big\} e^{\lambda_a (t-t_1)} - 1 \big\{ + D_{a2p} v_2 \big\} e^{\lambda_a (t-t_2)} - 1 \big\{ +$$
$$+ D_{b1p} v_1 \big\} e^{\lambda_b (t-t_1)} - 1 \big\{ + D_{b2p} v_2 \big\} e^{\lambda_b (t-t_2)} - 1 \big\{]^2 .$$

By subtraction, using (11) and (12), we arrive at

$$T - U = \frac{1}{2\lambda_a} [G_{a11} v_1^2 \big\} e^{\lambda_a (t-t_1)} - 1 \big\{^2 + 2 G_{a12} v_1 v_2 \big\} e^{\lambda_a (t-t_1)} - 1) \big\{ \ \big\} e^{\lambda_a (t-t_2)} - 1 \big\{ +$$
$$+ G_{a22}\, v_2^2 \big\} e^{\lambda_a (t-t_2)} - 1 \big\{^2] +$$
$$+ \frac{1}{2\lambda_b} [G_{b11}\, v_1^2 \big\} e^{\lambda_b (t-t_1)} - 1 \big\{^2 + 2 G_{b12} v_1 v_2 \big\} e^{\lambda_b (t-t_1)} - 1 \big\{ \ \big\} e^{\lambda_b (t-t_2)} - 1 \big\{ +$$
$$+ G_{b22}\, v_2^2 \big\} e^{\lambda_b (t-t_2)} - 1 \big\{^2] .$$

As is well known, the application of a variable voltage can be conceived as the successive application of incremental constant voltages. Therefore, if a constant voltage v, applied at $t = 0$ in some point A of a network at rest, gives rise, for $t > 0$ in some point B of the network, to a voltage or current $vf(t)$, then a variable voltage v, applied at $t = 0$ in the point

A of the network at rest, will give rise, for $t > 0$ in the point B, to a voltage or current $\int_0^t v'(\tau) f(t-\tau) \mathrm{d}\tau$. If variable voltages v_1 and v_2 are applied at $t = 0$ to the four-pole at rest, we can thus for $t = 0$ write instead of (10)

$$i_{Lm} = B_{a1m} \int_0^t v_1'(\tau) \}e^{\lambda_a(t-\tau)} - 1\{ \mathrm{d}\tau + B_{a2m} \int_0^t v_2'(\tau) \}e^{\lambda_a(t-\tau)} - 1\{ \mathrm{d}\tau +$$
$$+ B_{b1m} \int_0^t v_1'(\tau) \}e^{\lambda_b(t-\tau)} - 1\{ \mathrm{d}\tau + B_{b2m} \int_0^t v_2'(\tau) \}e^{\lambda_b(t-\tau)} - 1\{ \mathrm{d}\tau .$$

The B's are the same as in (10), but the time functions have changed. Similarly, in v_{Cp}, T, U, and $T-U$ the coefficients remain the same, and only the time functions are altered. This proves our statement in section 5 that also in the case of variable voltages $T-U$ does not depend on the particular network used for realizing a given admittance matrix. This result can be extended to networks with any number of terminal pairs and with admittance matrices having any number of poles.

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REFERENCES

1) H. W. Bode, Network analysis and feedback amplifier design, 1945, p. 130.
2) B. D. H. Tellegen, Philips Res. Rep. 3, 81, 1948.
3) O. Heaviside, Electrical Papers, Vol. II, p. 202.
4) O. Heaviside, Electrical Papers, vol. II, p. 412;
H. A. Lorentz, Proc. nat. Acad. Sciences 8, 333, 1922 = Collected Papers, Vol. III, p. 331. See also Balth. van der Pol, Physica, 's-Grav. 4, 585, 1937.