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A METHOD OF MEASURING SPECIFIC RESISTIVITY AND HALL EFFECT OF DISCS OF ARBITRARY SHAPE

by L. J. van der PAUW

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Summary
A method of measuring specific resistivity and Hall effect of flat samples of arbitrary shape is presented. The method is based upon a theorem which holds for a flat sample of arbitrary shape if the contacts are sufficiently small and located at the circumference of the sample. Furthermore, the sample must be singly connected, i.e., it should not have isolated holes.

Résumé
On présente une méthode pour mesurer la résistance spécifique et l'effet Hall d'un échantillon plat de forme quelconque. La méthode est fondée sur un théorème qui est appliquable si l'échantillon est plan-parallèle, si les contacts sont suffisamment petits et se trouvent à la périphérie de l'échantillon. Enfin l'échantillon doit être simplement connexe, c-à-d. sans trous isolés.

Zusammenfassung
Es wird eine Methode zur Messung des spezifischen Widerstandes und des Halleffektes einer planparallelen Probe willkürlicher Form angegeben. Die Methode gründet sich auf eine These, die anwendbar ist wenn die Kontakte genügend klein sind und sich am Rande der Probe befinden. Schließlich soll die Probe einfach zusammenhängend sein, d.h. sie darf keine Löcher haben.

1. Introduction

In many cases the specific resistivity and the Hall effect of a conducting material are measured by cutting a sample in the form of a bar. Current contacts A and B and voltage contacts C, D, E and F are attached to the bar as shown in fig. 1. The specific resistivity is then derived from the

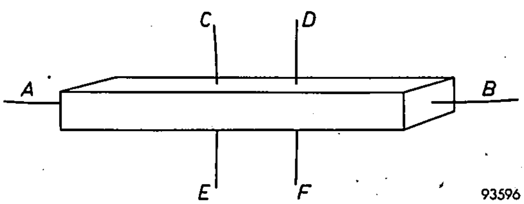

Fig. 1. The classical shape of a sample for measuring the specific resistivity and the Hall effect.

potential drop between the points C and D or E and F and from the dimensions of the sample. On the other hand, the Hall voltage can be measured between the points C and E or D and F. The current contacts must be far away from the points C, D, E and F in order to ensure that the lines of flow are sufficiently parallel and are not changed on application of a magnetic field.

For the measurement of the specific resistivity and Hall effect of semiconductors a more complicated shape of the sample has often to be used. A well-known example is the bridge-shaped sample shown in fig. 2. The

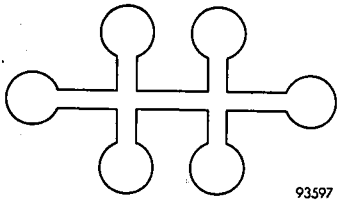

Fig. 2. The bridge-shaped sample, furnished with large areas for making low-ohmic contacts.

large areas at the ends have the task to provide low-ohmic contacts. Furthermore, when making these contacts a heat treatment is often necessary which in this case can be done without heating that part of the sample which is under measurement.

It will be shown that the specific resistivity and the Hall effect of a flat sample of arbitrary shape can be measured without knowing the current pattern if the following conditions are fulfilled:

(a) The contacts are at the circumference of the sample.
(b) The contacts are sufficiently small.
(c) The sample is homogeneous in thickness.
(d) The surface of the sample is singly connected, i.e., the sample does not have isolated holes.

2. A theorem which holds for a flat sample of arbitrary shape

We consider a flat sample of a conducting material of arbitrary shape with successive contacts A, B, C and D fixed on arbitrary places along the circumference such that the above-mentioned conditions (a) to (d) are fulfilled (see fig. 3). We define the resistance $R_{AB,CD}$ as the potential difference

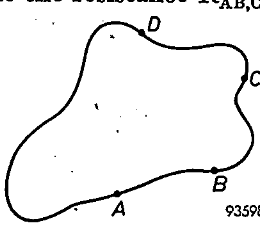

Fig. 3. A sample of arbitrary shape with four small contacts at arbitrary places along the circumference which, according to this paper, can be used to measure the specific resistivity and the Hall effect.

$V_D - V_C$ between the contacts D and C per unit current through the contacts A and B. The current enters the sample through the contact A and leaves it through the contact B. Similarly we define the resistance $R_{BC,DA}$. It will be shown that the following relation holds:

$$\exp(-\pi R_{AB,CD}\, d/\varrho) + \exp(-\pi R_{BC,DA}\, d/\varrho) = 1\,, \tag{1}$$

where ϱ is the specific resistance of the material and d is the thickness of the sample.

To prove eq. (1) we shall first show that it holds for a particular shape of the sample. The second step is to prove that if it holds for a particular shape it will hold for any shape. For our particular shape we choose a semi-infinite plane with contacts P, Q, R and S along its boundary, spaced at distances a, b and c respectively (see fig. 4). A current j enters the sample

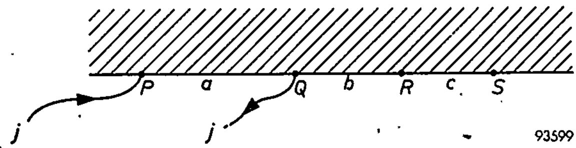

Fig. 4. A sample in the form of a semi-infinite plane with four contacts along its boundary for which eq. (1) is proved first.

at the contact P and leaves it at the contact Q. From elementary theory it follows that

$$V_S - V_R = \frac{j\varrho}{\pi d} \ln \frac{(a+b)(b+c)}{b(a+b+c)}.$$

Hence

$$R_{PQ,RS} = \frac{\varrho}{\pi d} \ln \frac{(a+b)(b+c)}{b(a+b+c)}. \tag{2}$$

In the same way, we have

$$R_{QR,SP} = \frac{\varrho}{\pi d} \ln \frac{(a+b)(b+c)}{ca}. \tag{3}$$

Moreover,

$$b(a+b+c) + ca = (a+b)(b+c)\,. \tag{4}$$

From the eqs (2), (3) and (4) eq. (1) follows immediately.

Using the same arguments it can also be shown that

$$R_{PQ,RS} = R_{RS,PQ}\,, \tag{5}$$

$$R_{QR,SP} = R_{SP,QR}\,, \tag{6}$$

$$R_{PR,QS} = R_{QS,PR}\,, \tag{7}$$

$$R_{PQ,SR} + R_{QR,SP} + R_{PR,QS} = 0\,. \tag{8}$$

The last four relations, however, are of a much more general nature than (1) and follow also from the reciprocity theorem of passive multipoles.

We shall now proceed with the second step and show that eq. (1) holds quite generally. To that end we make use of the well-known technique of conformal mapping of two-dimensional fields *). We assume that the semi-infinite sample considered above coincides with the upper part of the complex z-plane, where $z = x + iy$.

We introduce a function $w = f(z) = u(x,y) + iv(x,y)$, where u and v are both real functions of x and y. The function $f(z)$ is chosen in such a way that u represents the potential field in the sample. The functions u and v satisfy the Cauchy-Riemann relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{9}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{10}$$

If we now travel from an arbitrary point T_1 in the upper half-plane to another point T_2 in the upper half-plane (see fig. 5), the net current which

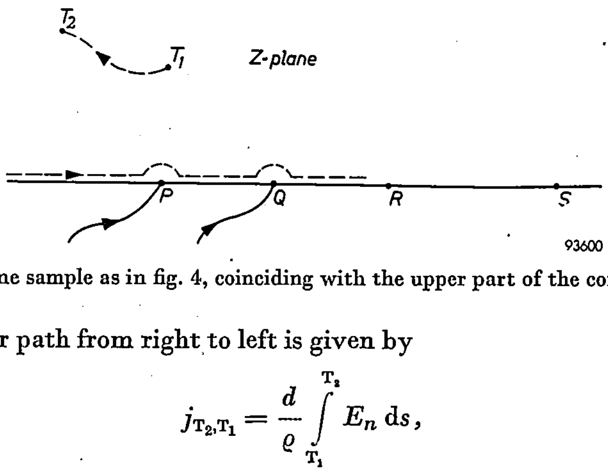

Fig. 5. The same sample as in fig. 4, coinciding with the upper part of the complex z-plane.

traverses our path from right to left is given by

$$j_{T_2,T_1} = \frac{d}{\varrho} \int_{T_1}^{T_2} E_n \, ds,$$

where E_n is the normal component of the field strength. This expression is readily verified to be equal to

$$j_{T_2,T_1} = \frac{d}{\varrho} \int_{T_1}^{T_2} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) = \frac{d}{\varrho} \int_{T_1}^{T_2} \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) = \frac{d}{\varrho} \left(v_{T_2} - v_{T_1} \right).$$

Hence if we travel along the real axis from $-\infty$ to $+\infty$ the value of v remains constant until we pass the point P. When passing the point P

*) L. V. Bewley, Two-dimensional fields in electrical engineering, The MacMillan Company, New York, 1948.

along a small half-circle in the upper half-plane the value of v will increase by $\varrho j/d$. Similarly when passing the point Q the value of v will decrease by $\varrho j/d$. We consider now a sample of arbitrary shape, lying in a different complex plane which we shall call the t-plane (see fig. 6), where $t = r + is$.

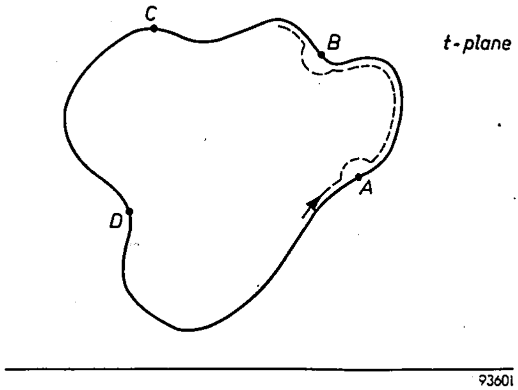

Fig. 6. A sample of arbitrary shape, lying in the complex t-plane.

By a well-known theorem, it is always possible to find an analytic function $t(z)$ such that the upper half-plane in the z-plane is mapped onto the sample in the t-plane. There are some restrictions as to the shape of the sample in the t-plane which are, however, not of physical interest. In particular, let A, B, C and D in the t-plane be the images of the points P, Q, R and S respectively in the z-plane. Furthermore, let $k(t) = l + im$ be identical with $f(z) = f(z(t)) = k(t)$. Hence by definition m remains constant when travelling in counter-clockwise direction along the boundary of the sample in the t-plane; it only increases by $\varrho j/d$ when passing the point A and it decreases by the same amount when passing the point B.

From the theory of conformal mapping it follows that if m in the t-plane is interpreted in the same way as v in the z-plane, then l will represent the potential field in the t-plane. Consequently if a current j' enters the sample at the contact A and leaves it at the contact B and if we choose $j'\varrho'/d' = j\varrho/d$, where ϱ' and d' are the specific resistivity and the thickness of the sample in the t-plane, then the voltage difference $V_D - V_C$ will be equal to the voltage difference $V_S - V_R$. Hence $(d/\varrho)\ R_{AB,CD}$ is invariant under conformal transformation. The same is true for $(d/\varrho)\ R_{BC,DA}$. From this it follows that eq. (1) is of general validity.

3. Practical applications

From the above section it follows that for measuring the specific resistivity of a flat sample it suffices to make four small contacts along its circumference and to measure the two resistances $R_{AB,CD}$ and $R_{BC,DA}$ (see fig. 3) and the thickness of the sample. Equation (1) determines uniquely the value of ϱ as a function of $R_{AB,CD}$, $R_{BC,DA}$ and d. In order to facilitate the solution of ϱ from eq. (1) we write it in the form

$$\varrho = \frac{\pi d}{\ln 2} \frac{(R_{AB,CD} + R_{BC,DA})}{2} f\left(\frac{R_{AB,CD}}{R_{BC,DA}}\right), \tag{11}$$

where f is a function of the ratio $R_{AB,CD}/R_{BC,DA}$ only and satisfies the relation

$$\frac{R_{AB,CD} - R_{BC,DA}}{R_{AB,CD} + R_{BC,DA}} = f \operatorname{arccosh} \left\{\frac{\exp(\ln 2/f)}{2}\right\}. \tag{12}$$

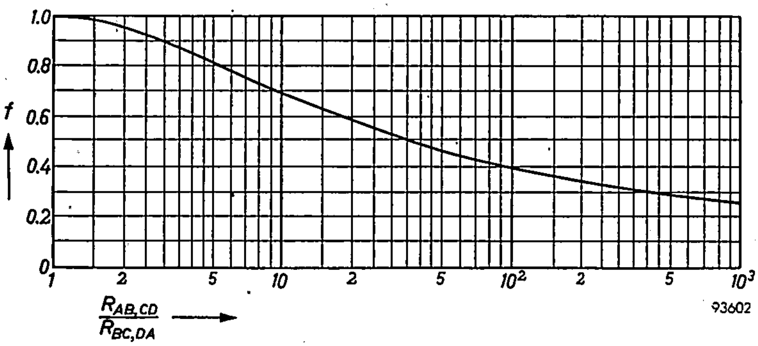

Fig. 7. The function f used for determining the specific resistivity of the sample, plotted as a function of $R_{AB,CD}/R_{BC,DA}$.

In fig. 7 a plot is given of f as a function of $R_{AB,CD}/R_{BC,DA}$. If $R_{AB,CD}$ and $R_{BC,DA}$ are almost equal, f can be approximated by the formula

$$f \approx 1 - \left(\frac{R_{AB,CD} - R_{BC,DA}}{R_{AB,CD} + R_{BC,DA}}\right)^2 \frac{\ln 2}{2} - \left(\frac{R_{AB,CD} - R_{BC,DA}}{R_{AB,CD} + R_{BC,DA}}\right)^4 \left\{\frac{(\ln 2)^2}{4} - \frac{(\ln 2)^3}{12}\right\}.$$

The Hall mobility can be determined by measuring the change of the resistance $R_{BD,AC}$ when a magnetic field is applied perpendicular to the sample. The Hall mobility is then given by

$$\mu_H = \frac{d}{B} \frac{\Delta R_{BD,AC}}{\varrho}, \tag{13}$$

where B is the magnetic induction and $\Delta R_{BD,AC}$ the change of the resistance $R_{BD,AC}$ due to the magnetic field.

Equation (13) is based upon the following argument: If we apply a magnetic field perpendicular to the sample the equations

$$\operatorname{div} \mathbf{j} = 0, \tag{14}$$

$$\operatorname{curl} \mathbf{j} = 0, \tag{15}$$

where $\mathbf{j}$ represents the current density, remain valid. Furthermore if the contacts are sufficiently small and at the circumference of the sample the outer lines of flow, which must follow the circumference of the sample, fully determine our boundary conditions. Hence the lines of flow do not change when a magnetic field is applied. However, the effect of the magnetic field on the electric potential is such that between two arbitrary points an additional potential difference ΔV is built up which is equal to

$$\Delta V = \frac{\mu_H B j \varrho}{d}, \tag{16}$$

where j is the current which passes between the two points. Equation (13) follows immediately from (16).

In order to estimate the order of magnitude of the error introduced if the contacts are of finite size and not at the circumference of the sample we derived an approximation formula for a few special cases. In all cases we assumed that the sample had the form of a circular disc with contacts spaced at angles of 90°. Furthermore we assumed that the area over which the contact is made is an equipotential area. We shall denote by $\Delta\varrho/\varrho$ and $\Delta\mu_H/\mu_H$ the relative errors introduced in the measurement of the specific resistivity and the Hall mobility, respectively.

In fig. 8*a* is presented the case in which one of the contacts is of finite length d; it is assumed to lie along the circumference of the sample. The

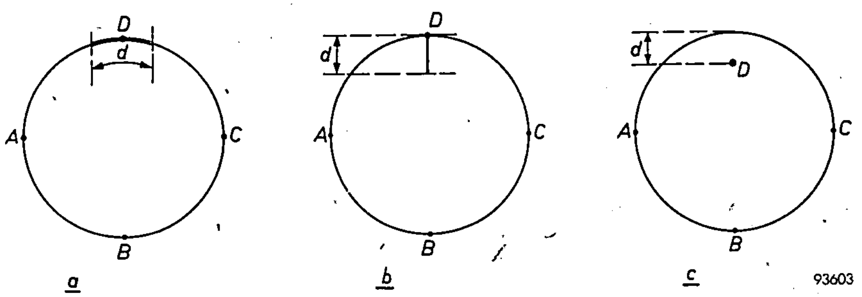

Fig. 8. Some special cases for which the error in the measurement of ϱ and μ_H due to the finite length or the finite distance to the circumference of one contact has been calculated.

other contacts are infinitely small and located at the circumference. The diameter of the sample will be denoted by D. In this case for a small value of d/D and of μB the following relations may be shown to hold:

$$\frac{\Delta\varrho}{\varrho} \approx \frac{-d^2}{16\,D^2 \ln 2}, \tag{14}$$

$$\frac{\Delta\mu_H}{\mu_H} \approx -\frac{2d}{\pi^2 D}, \tag{15}$$

In fig. 8b is shown the case in which the contact is made in the direction perpendicular to the circumference. In this case the error introduced will be as in the foregoing case, but with d twice as large:

$$\frac{\Delta\varrho}{\varrho} = \frac{-d^2}{4D^2 \ln 2}, \tag{16}$$

$$\frac{\Delta\mu_H}{\mu_H} = \frac{-4d}{\pi^2 D}. \tag{17}$$

Finally we consider the case in which one contact lies at a distance d from the circumference (see fig. 8c). In this case we obtain

$$\frac{\Delta\varrho}{\varrho} = -\frac{d^2}{2D^2 \ln 2}, \tag{18}$$

$$\frac{\Delta\mu_H}{\mu_H} = -\frac{2d}{\pi D}. \tag{19}$$

It can be shown that if more contacts have at the same time some of the above-mentioned defects the errors introduced are to a first approximation additive.

The influence of the contacts can be eliminated still further by using a "clover-shaped" sample, as shown in fig. 9. This sample has many advan-

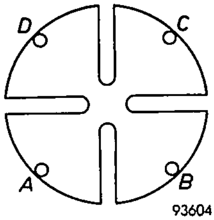

Fig. 9. The "clover-shaped" sample where the influence of the contacts has been reduced considerably.

tages compared with the bridge-shaped sample. It gives a relatively large Hall effect at the same amount of heat dissipation, which is of importance when measuring materials of low electric mobility. It has a greater mechanical strength and smaller samples can be measured which is of importance, for example, when measuring silicon crystals made by the floating-zone technique.

Eindhoven, September 1957

Note added in proof

In sec. 2 we derived a relation between the resistances $R_{AB,CD}$, $R_{BC,DA}$ and ϱ/d if all contacts are at the circumference of the sample and infinitely small. If the contacts are all of finite size there will be in general six independent finite resistances, for example the resistances $R_{AB,AB}$, $R_{AC,AC}$, $R_{AD,AD}$, $R_{BC,BC}$, $R_{BD,BD}$ and $R_{CD,CD}$. We assume that the contacts are areas of constant potential. It can be shown that, if the contacts are located at the circumference of the sample, also in this case there must be a relation between these six resistances and ϱ/d which determines ϱ/d uniquely as a function of these six resistances. If there is only one contact of finite size, A say, it can be shown that

$$\exp\left(\frac{\pi d}{\varrho} R_{AB,CD}\right) + \exp\left(\frac{\pi d}{\varrho} R_{BC,DA}\right) - \exp\left[\frac{\pi d}{\varrho}\left(R_{AB,CD} + R_{BC,DA}\right)\right] - \exp\left(\frac{2\pi d}{\varrho} R_{AB,DA}\right) = 0 .$$

The author is indebted to Dr C. J. Bouwkamp of this laboratory for pointing out to him that, if more than one contact is of finite size, the relation between the independent resistances and the specific resistivity of the sample involves elliptic or hyper-elliptic functions rather than elementary functions.

Professor Bouwkamp has also drawn the author's attention to a recent paper by Lampard *), who deals with the calculation of internal cross capacitances of cylinders under certain conditions of symmetry. Lampard's result can be generalized as follows. Let fig. 6 of this paper represent the cross-section of a cylindrical capacitor, cut into four parts insulated from one another at the points A, B, C and D. Let $C_{AB,CD}$ denote the internal cross capacitance of parts AB and CD, in electrostatic c.g.s. units per unit length of cylinder. Similarly, let $C_{BC,DA}$ denote the internal cross capacitance of BC and CD. Then we have

$$\exp(-4\,\pi^2\, C_{AB,CD}) + \exp(-4\,\pi^2\, C_{BC,DA}) = 1,$$

which is identical with eq. (1) of this paper except for the different physical interpretation.

In Lampard's case of symmetry, the two capacitances $C_{AB,CD}$ and $C_{BC,DA}$ are mutually equal, and hence are both equal to $(\ln 2)/4\pi^2$ independently of the size or shape of the cross-section, which is Lampard's theorem.

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*) D. G. Lampard, Proc. Instn elect. Engrs, Part C, Vol. 104, No. 6, Sept. 1957, p. 271.