ON WAVE PROPAGATION
IN BEAM-PLASMA SYSTEMS

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Summary
The dispersion equation of waves in a cylindrical waveguide completely
filled with a number of beams of charged particles and with an infinite
magnetic field in the direction of the axis is studied. A simple analytical
method is developed, giving the dispersion diagrams of the non-
growing waves in such systems. The frequency bands where growing
waves occur can then easily be found, because for each frequency the
number of propagation constants satisfying the dispersion relation must
be the same. The method described is also applicable to finding the
ranges where non-convective instabilities occur. Two numerical examples
of dispersion diagrams, calculated with the aid of a computer, for the
non-growing and for the growing waves in a waveguide filled with a
beam and a plasma are given, both for the case with and without colli-
sions of the plasma electrons. The method of the numerical calculation
is explained in the appendix. It is shown that the velocity of the wave
front propagating in an infinite beam-plasma system never exceeds the
beam velocity, even in regions of anomalous dispersion. The amount
of energy propagating in a beam-plasma system is calculated for the
case of growing waves in an infinite interaction region and appears to
be zero when collisions of the electrons are neglected. For the non-
growing waves in a finite system the power flow can be related to the
difference between the phase velocity and the group velocity.

Résumé
Etude en détail de l'équation de dispersion des ondes dans un guide
d'ondes entièrement rempli de faisceaux ou par un faisceau et un
plasma. L'on suppose un champ magnétique infini se propageant dans
la direction de l'axe. Une simple méthode analytique est exposée. Elle
donne les courbes de dispersion des ondes non-croissantes dans le
système en question. Les bandes de fréquences où se produisent les
ondes croissantes peuvent être trouvées aisément car, pour chaque
fréquence, le nombre de constantes de propagation satisfaits à la
relation de dispersion doit être le même. La méthode décrite peut
s'appliquer également pour trouver les gammes où peuvent se produire
des "non-convective instabilities". Deux exemples numériques calculés
de diagrammes de dispersion pour les ondes croissantes et non-croissan-
tes dans un guide d'ondes rempli par un faisceau et un plasma y figurent;
ils sont calculés à l'aide d'une calculatrice et cela pour les cas où se
produisent ou non des collisions d'électrons du plasma. La méthode
de calcul numérique est expliquée en annexe. Il est montré que la vitesse
du front d'ondes se propageant dans un système infini plasma-
faisceau, ne dépasse jamais la vitesse du faisceau, même dans les régions
ou la dispersion présente des anomalies. La quantité d'énergie répandue
dans un système plasma-faisceau est calculée pour le cas des ondes
s'étendant dans une région d'action réciproque infinie et se trouve être
nulle si l'on ne tient pas compte des collisions d'électrons. Pour les ondes
non-croissantes dans un système limité, le flux de puissance peut
être mis en relation avec la différence entre la vitesse de la phase et la
vitesse du groupe.
Zusammenfassung


1. Introduction

In recent years much attention has been paid to the interaction of an electron beam and a plasma. Owing to this interaction, growing waves can occur. These are waves that are described by \( \exp\{j(\omega t - \Gamma z)\} \) and where the propagation constant \( \Gamma = \beta + ja \) has a positive imaginary part, \( a \); \( \beta \) is the phase constant and \( a \) is the attenuation constant. The interest in these waves arises partly from the possibility of using them as the amplifying mechanism in microwave amplifiers or oscillators. The increment of the waves can be very high when the wave frequency is close to the resonance frequency of the plasma electrons. In an experiment performed by the authors, the growing rate was 19 dB/cm at 4000 Mc/s. The interaction length of the tube is 3 cm. An overall gain (including the losses of the coupling systems) of 30 dB was achieved. The plasma consisted of the positive column of a mercury discharge at \( 2 \times 10^{-3} \) torr. A 300-V, 2-mA beam was used. Using a feedback the device also runs as an oscillator.

The interaction of a beam and a plasma is also interesting because it provides a possible method to convert the d.c. energy of a beam into thermal energy in the plasma. Furthermore the occurrence of growing waves might be the explanation of several astronomical effects and of plasma oscillations in gas discharges.

The subject of this paper is mainly a method to study the dispersion relation of beam-plasma interaction in a very strong magnetic field parallel to the beam. The assumption of a strong magnetic field has been made since we then have...
one of the few cases in which an analytical study of the dispersion relation becomes relatively simple. The method described is not necessarily restricted to beam-plasma interaction but can also be applied to the dispersion relation of waves in a waveguide containing \( n \) different beams of charged particles. Furthermore it can be used equally well to look for the combinations of real values of the propagation constant \( \Gamma' \) and complex values of \( \omega \) that satisfy the dispersion relation; the associated nonconvective instabilities \(^4\) may be of practical importance.

2. Deduction of the dispersion relation

Let us study a circle-cylindrical waveguide containing \( n \) beams of charged particles having different d.c. velocities. The charge density of each beam is assumed to be homogeneous over the cross-section of the waveguide. A very strong static magnetic field, \( B_0 \), in the direction of the axis of the waveguide is assumed (fig. 1). Motions of the charged particles perpendicular to the axis of the waveguide are therefore suppressed and all static magnetic fields arising from d.c. currents of the beams can be neglected.

To solve the problem of wave propagation in such a system we use Maxwell's equations:

\[
\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t},
\]

\[
\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \sum_n j_n,
\]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic a.c. field strengths in the waveguide respectively and \( j_n \) the a.c. convection-current density of the \( n \)th beam; \( j_n \) is always in the \( z \)-direction due to \( B_0 \).

The linearized equation of motion for the particles of the \( n \)th beam in the \( z \)-direction is

\[
\eta_n \frac{dv_n}{dt} + v_n v_n = \frac{\partial v_n}{\partial t} + u_n \frac{\partial v_n}{\partial z} + v_n v_n,
\]

Fig. 1. Sketch of the physical situation and the coordinates used.
where \( \eta_n \) is the specific charge of the particles of the \( n \)th beam, \( v_n \) the a.c. velocity, \( u_n \) the d.c. velocity and \( v_n \) the collision frequency of the beam particles. We only have to consider the \( z \)-direction, as far as the equations of motion concerns, because of the strong \( B_0 \) field. Furthermore we have the continuity equation for each beam

\[
\frac{\partial j_n}{\partial z} = - \frac{\partial \rho_{1n}}{\partial t} \tag{4}
\]

where \( \rho_{1n} \) means the a.c. density, and where \( j_n \) is given by the current equation. The latter reads, using the small-signal assumption,

\[
j_n = \rho_{0n} v_n + \rho_{1n} u_n \tag{5}
\]

with \( \rho_{0n} \) being the d.c. space-charge density. From eqs (4) and (2), of course, we find

\[
\nabla E = \frac{1}{\varepsilon_0} \sum _n \rho_{1n}. \tag{4a}
\]

We assume solutions for which the dependence on \( t \) and \( z \) is given by the factor \( \exp \{ j(\omega t - \Gamma z) \} \) so that we can replace \( \partial / \partial t \) by \( j\omega \) and \( \partial / \partial z \) by \( -j\Gamma \); then the equations reduce to differential equations in \( r \) and \( \theta \) only. Taking the curl of eq. (1) and substituting it in eq. (2) we obtain an equation in \( E \) and \( j_n \). After applying the relation \( \nabla \times (\nabla \times a) = \nabla (\nabla a) - \Delta a \) and inserting eq. (4a) we express \( J_n \) and \( \rho_{1n} \) in terms of \( E_z \) by means of eqs (3), (4) and (5).

The \( z \)-component of the equation obtained is the wave equation of the \( E_z \) field propagating in the system concerned. This equation is a Bessel equation which reads in this case

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + T^2(\Gamma, \omega) E_z = 0, \tag{6a}
\]

where

\[
T^2(\Gamma, \omega) = (k^2 - \Gamma^2)(1 - \sum _n \frac{\omega_{pn}^2}{(\omega - \Gamma u_n)(\omega - \Gamma u_n - jv_n)}), \tag{7a}
\]

\( k^2 = \omega^2 \varepsilon_0 \mu_0 = \omega^2 / c^2 \) and \( \omega_{pn}^2 = \eta_n \rho_{0n} / \varepsilon_0 \) (\( \omega_{pn} \) is the plasma frequency of the \( n \)th beam). From eq. (6) we derive the following complete mode which is finite for \( r = 0 \):

\[
E_z = \hat{E}_{zpq} J_p \{ T(\Gamma, \omega) r \} e^{j\phi} e^{j(\omega t - \Gamma z)}. \tag{8}
\]

Since at the waveguide wall (\( r = R \)) the electric field tangential to the wall surface must be zero, the boundary condition is

\[
T(\Gamma, \omega) R = \xi_{pq}, \tag{7b}
\]

where \( \xi_{pq} \) is the \( q \)-th root of the \( p \)-th Bessel function.
For one special mode (specified by the values of the integers $p$ and $q$) $T(\Gamma, \omega)$ is given by eq. (7b) so that for every value of $\omega$, $\Gamma$ is given by eq. (7a). The dispersion relation for the waves is therefore eq. (7a) with $T$ given by eq. (7b).

Using Maxwell's equations the $E_r$ and $H_\theta$ components can be derived in the usual way from the $E_z$ component. The field configuration is comparable to the TE-waves in an empty waveguide. In a similar way the TM-waves can be deduced. They are identical with the TM-waves in an empty guide because these waves, having no $E_z$ component, cannot interact with the charge carriers of the system, on account of the high d.c. magnetic field $B_0$.

3. Analytical study of the dispersion equation

For the analytical study we assume $\nu_n$ to be zero. In that case eq. (7) is of the $(2n + 2)$th order (supposing $u_n \neq 0$) with real coefficients. We now introduce $\Gamma = \omega(B + jA)$, where $B$ and $A$ are still unknown real functions of the real variable $\omega$. This substitution is found to lead to a mathematical separation of the growing and the non-growing waves of the system. It will be shown that the dispersion diagram of the latter ones can be derived in a relatively simple manner. We note that $\omega B$ is equal to the phase constant, $\beta$, and $\omega A$ is equal to the attenuation constant, $\alpha$, of the waves considered. Substituting this in eq. (7a) and solving for $\omega^2$ yields

$$
\omega^2 = \sum_n \frac{(1 - Bu_n)^2 - A^2 u_n^2}{\{(1 - Bu_n)^2 + A^2 u_n^2\}^2} \omega_{pn}^2 + \frac{1}{c^2 + A^2 - B^2 - 2jAB} + T^2 \frac{\left(\frac{1}{c^2} + A^2 - B^2\right)^2 + 4 A^2 B^2}{c^2 + A^2 + B^2},
\tag{9}
$$

where $T$ is the quantity given by eq. (7b) if we consider one special mode. By taking real and imaginary parts of (9) and because $\omega$ was assumed to be real this equation can easily be split up into

$$
\omega^2 = \sum_n \frac{(1 - Bu_n)^2 - A^2 u_n^2}{\{(1 - Bu_n)^2 + A^2 u_n^2\}^2} \omega_{pn}^2 + \frac{T^2 \left(\frac{1}{c^2} + A^2 - B^2\right)}{\left(\frac{1}{c^2} + A^2 - B^2\right)^2 + 4 A^2 B^2} \tag{10}
$$

and

$$
0 = \sum_n \frac{A u_n (1 - Bu_n)}{\{(1 - Bu_n)^2 + A^2 u_n^2\}^2} \omega_{pn}^2 + \frac{AB}{\left(\frac{1}{c^2} + A^2 - B^2\right)^2 + 4 A^2 B^2} T^2. \tag{11}
$$
The latter equation obviously has a solution $A = 0$. This essentially means that non-growing waves exist in the system studied. We first consider the dispersion of these waves by substituting $A = 0$ into eq. (10). This yields

$$\omega^2 = \sum_{n} \frac{\omega_p n^2}{(1 - Bu)^2} + \frac{T^2 c^2}{1 - B^2 c^2}.$$  \hspace{1cm} (12)

This equation, consisting of the sum of simple terms, can easily be represented in an $\omega^2$-$B$-diagram. After transformation back to a $\beta$-$\omega$-diagram the dispersion diagram of all possible non-growing waves in the guide studied is obtained. Since the dispersion relation is of the $(2n + 2)$th order there must exist $2n + 2$ values of $\Gamma$ for each value of the radial frequency $\omega$. So a glance at the dispersion diagram for the non-growing waves gives us the frequency ranges in which necessarily complex propagation constants must occur, since in these ranges the number of non-growing waves is less than $2n + 2$.

4. Application to electron-beam-plasma interaction

In the case of the interaction of a single electron beam and a plasma, eq. (7) reduces to a biquadratic equation in $\Gamma$ and for eq. (12) we have, since the drift velocity of the plasma is zero,

$$\omega^2 = \omega_p^2 + \frac{\omega_p^2}{(1 - B u)^2} + \frac{T^2 c^2}{1 - B^2 c^2};$$  \hspace{1cm} (13)

$\omega_b$ and $\omega_p$ are the plasma frequencies of the beam and of the plasma, respectively, and $u$ is the drift velocity of the beam. The contribution of the ions of the plasma is neglected because their plasma frequency is very small in comparison to the electron plasma frequency.

Relation (13) holds for the non-growing waves in a wave-guide filled with
Fig. 3a. Graph of eq. (13) derived from the addition of the functions plotted in fig. 2.
Fig. 3b. Phase characteristics, $\beta$, of the waves resulting from interaction of an electron beam and a plasma in a waveguide. Solid lines: non-attenuated waves derived from fig. 3a. Dashed lines: growing and attenuating waves (two coincident solutions).
Fig. 3c. Attenuation constant, $\alpha$, of the waves.
a beam and a plasma. Its right-hand side consists of very simple contributions which are plotted in fig. 2 as a function of $B$.

Depending on the values of $T$, $u$, $\omega_p$ and $\omega_b$ the addition of the three right-hand terms of eq. (13) results in 4 different cases, characterized by the inequalities in the first column of fig. 3 *). The corresponding dependence of $\omega^2$ on $B$ is given in the second column of fig. 3. We only have to consider the region $\omega^2 > 0$ of the $\omega^2$-$B$-diagrams from which we derive the dispersion diagrams. The corresponding $\beta$-functions are given in the third column of fig. 3 (solid lines). The transformation of an $\omega^2$-$B$-diagram into a $\beta$-$\omega$-diagram is characterized by the fact that a curve $B = \text{constant}$ in the $\omega^2$-$B$-diagram corresponds to a straight line through the origin in the $\beta$-$\omega$ diagram, its directional tangent being $B$. In this way we have found the real solutions for $B$, using eq.(13).

Since the dispersion equation is biquadratic in $T$ we must find 4 values of $T$ for each $\omega$. So, in the regions indicated as I and II in the diagrams of fig. 3 we still lack two complex or imaginary solutions. If the regions I and II are bounded by an extreme value the missing solutions have to approach this value. Therefore it is of interest to calculate the coordinates of the extreme values in the $\omega^2$-$B$-diagram. These points give, after transformation to the $\beta$-$\omega$-diagram, boundaries of the regions where complex propagation constants occur. If the boundary is not given by such an extreme value it is the line $\omega = \omega_p$.

Differentiation of eq. (13) and setting to zero leads to

$$\frac{\partial \omega^2}{\partial B} = \frac{2 u \omega_b^2}{(1-Bu)^3} + \frac{2 T^2c^4B}{(1-B^2c^2)^2} = 0,$$

(14)

which holds for the extreme values. Inside the range $0 < B < 1/u$ eq. (14) has no roots, since both terms are positive. Due to the term $T^2c^2(1-B^2c^2)^{-1}$ in eq. (13) there must be one minimum in the range $-1/c < B < 0$. Since eq. (14) is biquadratic in $B$ and has real coefficients there must be either one or three solutions of this equation outside the range $-1/c < B < 1/u$. A glance at figs 2 and 3 convinces us that there is only one.

We first consider the minimum in the branch between $-1/c$ and $1/c$ in the $\omega^2$-$B$-diagram; the value of $B$ for which this minimum occurs is $B_{c0}$. After transformation to the $\beta$-$\omega$-diagram this branch appears between two curves through the origin and with directional tangents equal to $1/c$ and $-1/c$, respectively. This branch is not symmetrical with regard to the $\omega$-axis, as shown in fig. 3, third column. At $B = \beta = 0$ starts a small region with a backward wave. The displacement of $B_{c0}$ from $B = 0$, however, is of the order of $u^2/c^2$ and is therefore very small.

For the other extreme value we find approximately, when we assume $B^2 \gg 1/c^2$

*) The second inequality in the cases 3 and 4 of the first column of fig. 3 follows from eq. (15b).
\[ B_m u \approx \left(1 - \frac{\omega_b}{T u}\right)^{-1} \]  
(15)

and

\[ \omega_m^2 \approx \omega_p^2 - T^2 u^2 \left(1 - \frac{\omega_b}{T u}\right)^\frac{3}{2} \]  
(15b)

Especially if

\[ \omega_p^2 \leq T^2 u^2 \left(1 - \frac{\omega_b}{T u}\right)^\frac{3}{2} \]

the range II in which a complex solution exists is missing, i.e. the electron beam does not interact with the plasma such that growing waves arise for \( \omega < \omega_p \).

If one of the regions with complex solutions, I or II, is limited by the asymptote \( \omega = \omega_p \) it is shown in the appendix that the complex solutions \( \Gamma = \beta + j\alpha \) behaves as follows:

\[ \lim_{\omega \to \omega_p} \beta = \beta_p = \frac{\omega_b^2 \omega_p}{u(\omega_p^2 - T^2 u^2)} \]  
(16)

\[ \lim_{\omega \to \omega_p} \alpha = \pm \infty \]  
(17)

Furthermore the complex solutions must be complex conjugate as long as we neglect collisions.

From this information the complex solutions can be sketched in the diagrams of fig. 3, third column (broken lines). The corresponding values of \( \alpha \) are given in fig. 3, fourth column. A more-careful study of the growing waves is given in the appendix where also the case \( \nu \neq 0 \) will be considered. Results of numerical calculations are given in figs 4a and 4b.

Fig. 4a. Example of a dispersion diagram of an electron beam and a plasma in a waveguide calculated with an IBM-650 computer.
5. Discussion of the dispersion diagrams for beam-plasma interaction

In order to discuss the nature of the waves we first determine the dispersion diagram of beam-plasma interaction in an infinite interaction region. In this case we have \( T = 0 \), as follows from eq. (7b), taking \( R = \infty \). So, from eq. (7a) applied to beam-plasma interaction, we then find

\[
(k^2 - \Gamma^2) \left( 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - \Gamma \omega)^2} \right) = 0. \tag{18}
\]

Obviously we now have two independent types of solution. One, \( k^2 - \Gamma^2 = 0 \) belongs to purely transverse waves not interacting with any charge carrier at all (owing to the infinite magnetic field in the direction of propagation, interaction with transverse electromagnetic fields cannot exist). To show this mathematically one should start the solution of the basic equations with a plane TEM wave directly. The other type belongs to a purely longitudinal space-charge wave. In a bounded system these two types of wave are coupled. The solution for the longitudinal wave is

\[
\Gamma = \frac{\omega}{u} \pm \frac{\omega_b}{u} \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{-\frac{1}{2}}. \tag{19}
\]

The dispersion diagram of these waves is given in fig. 5a. When we also take into account the collisions of the plasma electrons, characterized by the collision frequency \( \nu \), a straight-forward calculation yields the following extension of eq. (19):

\[
\Gamma = \frac{\omega}{u} \pm \frac{\omega_b}{u} \left( 1 - \frac{\omega_p^2}{\omega(\omega - j\nu)} \right)^{-\frac{1}{2}}. \tag{19a}
\]
This equation is plotted in fig. 5b. The maximum value of the attenuation constant at \( \omega = \omega_p \) here becomes, when \( \omega_p \gg \nu \),

\[
a_{\text{max}} = \frac{\omega_b}{u} \sqrt{\frac{\omega_p}{2\nu}}.
\]

The dispersion equation for TE modes in an empty waveguide is, in view of eq. (7a) for \( \omega_{pn} = 0 \),

\[
T^2 + \Gamma^2 - k^2 = 0.
\] (20)

The corresponding dispersion diagram is shown in fig. 6. Comparing fig. 3 and 6 reveals the perturbed "empty waveguide mode" in a waveguide filled with a beam and a plasma. Furthermore in figs 3 and 5 the two "beam waves" perturbed by the plasma are found and the interaction region II also occurs in both figures. The complex solution in region I is the cut-off mode of the waveguide, perturbed by the presence of the beam and plasma electrons.
Fig. 6. Dispersion diagram of the waves in an empty guide. When $\omega > Tc$ the attenuation constant is zero.

We now arrive at two more points to discuss, namely the energy velocity and the amount of energy propagating along the beam-plasma system.

5.1. Energy velocity and wavefront velocity

In this section we shall restrict ourselves to the longitudinal waves in an infinite system ($T = 0$). Applying the definition of group velocity, $v_g = \omega / \beta$, to a beam-plasma system it appears that the slow beam-wave has a region where $v_g > c$ (cf. fig. 5). Apparently the formal definition breaks down here and the group velocity cannot be equal to the energy velocity. This difficulty has been met also in the discussion of light velocity in anomalous dispersive media 5).

We solve this difficulty by calculating the wave-front velocity of a disturbance propagating along the system. Such a disturbance is usually built up of a large number of travelling waves so that for this problem we cannot make the assumption of a single travelling wave, as is done elsewhere in this paper.

We therefore consider the equation which is obtained from eqs (1) to (5) for the longitudinal waves in an unbounded system (no $r$ and $\theta$ dependence of the field components, the space charge density and the currents) without making the assumption that the solution has the form $\exp\{j(\omega t - \Gamma z)\}$:

\[
\frac{\partial^4 E_z}{\partial t^4} + 2u \frac{\partial^4 E_z}{\partial t^3 \partial z} + u^2 \frac{\partial^4 E_z}{\partial t^2 \partial z^2} + (\omega_p^2 + \omega_b^2) \frac{\partial^2 E_z}{\partial t^2} + \\
+ 2u \omega_p^2 \frac{\partial^2 E_z}{\partial z \partial t} + u^2 \omega_p^2 \frac{\partial^2 E_z}{\partial z^2} = \phi(z, t) .
\tag{21}
\]

$\phi(z, t)$ is some excitation function which only differs from zero at possible
sources. We shall assume an excitation function $\phi$ of the form $\delta(z)\delta(t)$ ($\delta =$ Dirac function). Applying the two-sided Laplace transformation

$$F(p, q) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dt \ e^{-pt - qz} E_z(t, z),$$

(22)

eq (z) \bar{E}(t, z),

(22)

eq. (21) transforms into

$$pq = F(pq) \{p^4 + 2u p^3 q + u^2 p^2 q^2 + (\omega_p^2 + \omega_b^2) p^2 + 2u \omega_p^2 pq + u^2 \omega_p^2 q^2\}.$$

This involves the following solution of eq. (21) in the "image" space:

$$F(pq) = \frac{j}{2\omega_b u \sqrt{p^2 + \omega_p^2}} \left( \frac{q}{q + \frac{p}{u} + j \frac{p}{u'}} - \frac{q}{q - \frac{p}{u} - j \frac{p}{u'}} \right),$$

(23)

where

$$u' = \frac{u \sqrt{p^2 + \omega_p^2}}{\omega_b}.$$

We first transform this solution back to the $p-z$ space, remembering that $e^{-\omega z} U(z)$ constitutes the Laplace transform of $q(q + a)^{-1}$ for $\text{Re}(q + a) > 0$ ($U(z)$ is the unit step function, which is unity for $z > 0$ and zero for $z < 0$). We find

$$G(pz) = \frac{e^{-pz/u}}{u \omega_b \sqrt{p^2 + \omega_p^2}} U(z).$$

Then we transform back to the $z-t$ space and obtain for the corresponding inversion integral:

$$E_z(t, z) = \frac{1}{\omega_b u} \int_{t-z}^{t+z} U(z) \frac{e \left( \frac{t-z}{\omega} \right)}{p \sqrt{p^2 + \omega_p^2}} \sin \frac{pz}{u'} \ dp,$$

(24)

with $\xi > 0$.

Singularities of the integrand are at $p = \pm j \omega_p$. Furthermore $\sin \left( \frac{pz}{u'} \right) U(z) \left\{ p \sqrt{p^2 + \omega_p^2} \right\}^{-1}$ becomes zero for $p \to \infty$.

When $t < z/u$ the path of integration can be closed with a semi-circle through the right half of the complex plane. Using Jordan's lemma and the fact that there are no singularities enclosed by the contour of integration we conclude that the integral (24) is zero. When $t > u/z$, in order to apply Jordan's lemma, we must close the integration path with a semi-circle through the left halfplane and then the contour of integration always contains the two singularities. The conclusion is that the wave front of a disturbance propagates at most with the beam velocity, also at points where $\partial \beta/\partial \omega = 0$. Evaluation of the integral
(24) gives the wave form, but the very artificial input conditions make the solution of no practical use.

5.2. Amount of energy propagating along the system

We first consider the unbounded system. Since in this case \( E_r = H_\theta = 0 \), we have for the a.c. power flow \(^7\) through a unit surface element

\[
P = \frac{1}{4} \Re(Vj_\omega^*) = \frac{\frac{1}{2} u \omega \varepsilon \omega_p^2 e^{\frac{2\pi}{\omega} T} \hat{E}_z \hat{E}_z^* }{|\omega - \Gamma u|^4} \Re(\omega - \Gamma u),
\]

where \( V \) is the kinetic potential (defined by \( V = \frac{u v_b}{\eta} \), \( v_b \) is the a.c. beam velocity), \( j_\omega^* \) is the complex conjugate of \( j_\omega \), and \( \Gamma \) is given by eq. (19). Substitution of the value of \( \Gamma \) reveals that, when \( \omega < \omega_p \), the power flow is always zero, since \( \omega - \Gamma u \) is imaginary. Physically this means that no energy is needed for the growth of the a.c. modulation in an infinite beam-plasma system when no collisions occur. This resembles the ideal klystron case where a modulation can be put on a beam without giving energy to that beam. Only if we want to take off the modulation from the beam by passing it through some demodulation system do we increase the slow beam wave and therefore then extract energy from the d.c. energy of the beam. When collisions are taken into account the power-flow density \( P \neq 0 \), since the value of \( \Gamma \) obtained from eq. (19a), where \( \omega < \omega_p \), must be substituted in eq. (25). Then energy must be delivered to the plasma in order to sustain the oscillations. In that case the wave with the lowest phase velocity is the growing wave and energy is converted from the d.c. energy of the beam into random energy of the plasma.

In the case of a bounded interaction region \((T \neq 0)\) the electromagnetic term must also be taken into account. The power flow in one mode then becomes \(^7\)

\[
P_I = \int \frac{1}{4} \Re (Vj_\omega^* + E \times H^*) \, ds,
\]

where \( S \) is a cross-section of the waveguide and \( P_I \) is the total a.c. power flow in the waveguide. From Maxwell’s equations we calculate for the zero-order mode:

\[
E_r = \hat{E}_z \frac{j \Gamma}{k^2 - \Gamma^2} T J_1(\Gamma r) e^{i(\omega t - \Gamma z)},
\]

\[
H_\theta = \hat{E}_z \frac{j \omega \varepsilon_0}{k^2 - \Gamma^2} T J_1(\Gamma r) e^{i(\omega t - \Gamma z)}.
\]

If we take the products \( Vj_\omega^* \) and \( E_r H_\theta^* \) in every point of a cross-section of the waveguide and next integrate over this cross-section we arrive at *)

*) For higher-order modes we find the same expression except for the factor \( J_1^2(\Gamma r) \) which must be replaced by \(-J_{p+1}(\Gamma r)J_{p-1}(\Gamma r)\) for the \( p \)th mode.
\[ P_t = \frac{\pi \omega \epsilon_0 R^2 J_1^2(TR)}{2} e^{2\alpha} \text{Re} \left\{ \frac{T^2 \Gamma}{|k^2 - \Gamma|^2} + \frac{(\omega - \Gamma u) \omega_0^2}{|\omega - \Gamma u|^4} \right\} \hat{E}_z \hat{E}_z^*. \] (28)

For the non-growing waves we can substitute \( \Gamma = \omega B \) so as to obtain
\[ P_t = \frac{\pi R^2 \epsilon_0}{2 \omega^2} J_1^2(TR) \left\{ \frac{T^2 B c^4}{(1 - B^2 c^2)} + \frac{u \omega_0^2}{(1 - Bu)^3} \right\} |\hat{E}_z|^2 \]
or, in view of eq. (14),
\[ P_t = \frac{\pi R^2 \epsilon_0}{4 \omega} J_1^2(TR) \frac{\partial \omega^2}{\partial B} |\hat{E}_z|^2. \] (29)

This means that all at points where the complex solutions start (the cut-off points) the power flow is zero since \( \partial \omega^2/\partial B \) then vanishes. For the cut-off frequency of an empty waveguide this is a well-known fact.

The energy flow of the non-growing waves is proportional to the slope of the curve in fig. 3, second column, and from this figure it can easily be seen which term of eq. (13) determines the slope in a particular range of \( B \) and \( \omega \). This is a measure of how the energy is divided between kinetic and electro-magnetic energy. It is of interest to remark that for non-growing waves, \( \Gamma = \beta \),
\[ \frac{\partial \omega^2}{\partial B} = \frac{2 \omega^2}{\partial \beta - \beta \omega}. \]

Substitution of this expression into eq. (29) gives a relation between the amount of power carried by the non-growing waves and the group and phase velocities. A similar expression has been given by Beam \(^8\) for cyclotron waves on an electron beam.

For the growing waves up to now we could not find such a simple relationship between the amount of energy propagating along the system studied and the propagation constant \( \Gamma \). Equation (28), however, holds also for the growing waves and might be useful in a further study of these waves. A general study of the power flow in a waveguide containing \( n \) beams of charged particles leads to similar results.

6. Double-stream interaction

Applying the theory to the case of interaction between two electron streams without a plasma the dispersion equation becomes
\[ T^2 = (k^2 - \Gamma^2) \left( 1 - \frac{\omega p_1^2}{(\omega - \Gamma u_1)^2} - \frac{\omega p_2^2}{(\omega - \Gamma u_2)^2} \right) \] (30)
and eq. (12) now reads
\[ \omega^2 = \frac{T^2 c^2}{1 - B^2 c^2} + \frac{\omega p_1^2}{(1 - Bu_1)^2} + \frac{\omega p_2^2}{(1 - Bu_2)^2}. \] (31)
where $u_1$ and $u_2$ may be positive or negative. Practical applications are the double-stream interaction and the interaction of the electron and the ion stream in a gas discharge.

Figure 7 gives an example of double-stream interaction for streams in the same direction. As has already been mentioned, also in this case the energy flow of the non-growing waves, $P_t$, can be written as in eq. (29). In this case, however, $\omega^2$ is given by eq. (31). Furthermore, also in this case, depending on the values of $T$, $\omega_{pn}$ and $u_n$, different forms of dispersion diagrams occur, just as in the case for beam-plasma interaction (cf. fig. 3).

The fact that the method is applicable to the general case of $n$ streams of
charged particles suggests that it is also applicable to the case of a velocity
distribution. The dispersion relation of beam-plasma interaction with a non-
zero temperature of the plasma, in the approximation of Bohm and Gross 9),
becomes *

\[ T^2 = (k^2 - r^2) \left( 1 - \frac{\omega_p^2}{\omega^2 - v_T^2 \Gamma^2} - \frac{\omega_b^2}{(\omega - \Gamma u)^2} \right), \]

where \( v_T = 3 kT/m \).

Putting again \( \Gamma = \omega(B+jA) \), the equivalent of eq. (12) for this case becomes

\[ \omega^2 = \frac{T^2 c^2}{1 - B^2 c^2} + \frac{\omega_b^2}{(1 - Bu)^2} + \frac{\omega_p^2}{1 - v_T^2 B^2}, \]

which can be studied in a manner similar to the method applied to eq. (13) (cf. fig. 2).

7. Other solutions of the dispersion equation

The dispersion relation may also be fulfilled by a complex \( \omega \) at a given real
value of \( \Gamma = \beta \). In a completely analogous way as we used in sec. 3., we

\[ \text{Fig. 8. The non-convective instability of double-stream interaction in a waveguide.} \]

*) The phase velocities of the waves are outside the range of velocities covered by the velocity distribution.
can find these solutions by substituting \( \omega = \beta (v' + jv'') \) instead of
\[
\Gamma = \omega (B + jA). \tag{34}
\]
We arrive at an equation which is similar to eq. (12) which gives in this case all waves with \( v'' = 0 \):
\[
\beta^2 = \frac{T^2 c^2}{v'^2 - c^2} + \sum_n \frac{\omega_{pn}^2}{(v' - u_n)^2}. \tag{35}
\]
We can now draw a \( \beta-v' \)-diagram which gives us all solutions with real \( \omega \) (of course these solutions are the same as those obtained from eq. (12)). The missing solutions, which must be complex conjugate, can then be sketched in the \( \omega-\beta \)-diagram. There is one difference: the solutions of real \( \beta \) and complex \( \omega \) always start at a point where \( \delta \beta / \delta \omega = 0 \). In this way the instabilities can easily be found. In fig. 8 we give as an example the solution for double-stream interaction. Again for different parameters \( \omega_{pn}, u_n \) and \( T \), different forms of dispersion diagrams can be obtained. For a classification of these kinds of waves the reader is referred to ref. 4.

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Appendix

Numerical calculations have been performed for the case of beam-plasma interaction. The dispersion equation for this case, taking collisions of the plasma electrons with the neutral atoms into account, is given by
\[
T^2 = (k^2 - \Gamma^2) \left\{ 1 - \frac{\omega_p^2}{\omega(\omega - jv)} - \frac{\omega_b^2}{(\omega - \Gamma u)^2} \right\}. \tag{A,1}
\]
This is an algebraic equation of the fourth degree in \( \Gamma \) or \( \omega \). So for every real value of \( \omega \) four solutions of \( \Gamma \), which are in general complex, occur.

Substitution of \( \Gamma = \omega (B + jA) \) in eq. (A,1) and separation of the real and imaginary parts yields (cf. eqs (9), (10) and (11)):
\[
\omega^2 = \frac{\omega_p^2 \omega^2}{\omega^2 + v^2} + \omega_b^2 \frac{(1 - uB)^2 - u^2 A^2}{(1 - uB)^2 + u^2 A^2} + T^2 c^2 \frac{1 + c^2 A^2 - c^2 B^2}{(1 + c^2 A^2 - c^2 B^2)^2 + 4c^4 A^2 B^2}, \tag{A,2}
\]
\[
0 = \frac{\omega_p^2 \omega \nu}{\omega^2 + v^2} + \omega_b^2 \frac{2uA(1 - uB)}{(1 - uB)^2 + u^2 A^2} + T^2 c^2 \frac{2c^3 AB}{(1 + c^2 A^2 - c^2 B^2)^2 + 4c^4 A^2 B^2}. \tag{A,3}
\]
We first restrict ourselves to the case of \( \nu = 0 \). Then there are real solutions for \( \Gamma \), since eq. (A,3) is solved for \( A = 0 \). Substitution of this solution in eq. (A,2)
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gives eq. (13), which equation has been discussed in sec. 4. For every value of \( B \) there is one value of \( \omega^2 \). If the latter value is positive the values of \( B \) and \( \omega^2 \) yield a set of values of \( \Gamma \) and \( \omega \), which are the real solutions of eq. (A,1).

Since for every value of \( \omega \) four values of \( \Gamma \) exist, one can directly find in which frequency bands the propagation constant must be complex. For these complex values of \( \Gamma \) eq. (A,2) can be rewritten in the form

\[
\omega_b^2 u(1-uB) = T^2 c^4 B \lambda,
\]

\[
0 = \lambda \{(1 + c^2 A^2 - c^2 B^2)^2 + 4 c^4 A^2 B^2\} + \{(1-uB)^2 + A^2 u^2\}^2. \tag{A,4}
\]

For every value of \( B \) the first part of eq. (A,4) yields a value of \( \lambda \). The second part in eq. (A,4) is a quadratic equation in \( A^2 \), which therefore gives directly the corresponding value of \( A \) (only the positive roots can be used). Finally, substitution of \( A \) and \( B \) into eq. (A,2) gives the value of \( \omega \).

When \( \nu \neq 0 \) the calculations cannot be performed in such a simple manner. The equations are now solved by using an iterative method, where the solutions for the case \( \nu = 0 \) can be used as a first approximation.

From the theory of sec. 4 we know the values of \( \Gamma \) and \( \omega \) at the boundaries of the regions where complex solutions for \( \Gamma \) occur (cf. regions I and II of fig. 3). Only at the asymptote \( \omega = \omega_p \) in the \( \beta \)-\( \omega \)-diagram do we not yet know the limiting values. In order to calculate these values for the case \( \nu = 0 \) we assume \( A = \pm \infty \) to be a solution of eq. (A,4). In that case the coefficient of \( A^4 \) must be zero, which yields \( \lambda = -u^4/c^4 \), or in view of the first part of eq. (A,4),

\[
B_p = \frac{\omega_b^2}{u(\omega_b^2-T^2 u^2)}. \tag{A,5}
\]

When \( A = \pm \infty \) we observe from eq. (A,2) that \( \omega = \omega_p \). In this manner we have obtained the value of \( \beta_p = \omega_p B_p \) in fig. 3, which value is very useful if only a sketch of the dispersion diagram is needed. The result of this calculation has already been given in eqs (16) and (17).

REFERENCES