THE MAXIMUM NUMBER OF PARAMETERS OF N-PORTS OF VARIOUS CLASSES

by V. BELEVITCH

Abstract
The maximum number of parameters is computed for the following classes of \( n \)-ports of degree \( m \) : (1) general, (2) reciprocal, (3) lossless and (4) lossless reciprocal. The results were known for classes (1) and (2) but a simpler proof is presented. The results for the other two classes are new.

We wish to compute the maximum number of independent parameters characterizing an \( n \)-port of degree \( m \), taking into account various possible restrictions such as reciprocity or losslessness. The results are summarized below:

\[
\begin{align*}
\text{general } n\text{-port: } & A_{nm} = n(2m + n), \\
\text{reciprocal } n\text{-port: } & B_{nm} = (n + 1)(m + n/2), \\
\text{lossless non-reciprocal } n\text{-port: } & C_{nm} = n[m + (n - 1)/2], \\
\text{lossless reciprocal } n\text{-port: } & D_{nm} = \frac{n+1}{2} \left(m + \frac{n-1}{2}\right) + \frac{(-1)^m}{8} [1 + (-1)^n].
\end{align*}
\]

The value of \( B_{nm} \) has been computed by Tellegen 1) and results independently from a synthesis process of the Brune type described by Belevitch 2). The value of \( A_{nm} \) has been computed by Belevitch 3) and has been similarly confirmed 2). The values of \( C_{nm} \) and \( D_{nm} \) are new and will be established below. The particular value \( C_{1,m} = D_{1,m} = m \) is well known from Foster’s theorem, the value \( C_{2,m} = 2m + 1 \) was established by Tellegen 4) and the result

\[
D_{2,m} = \begin{cases} 
1 + 3m/2 & \text{for } m \text{ even} \\
(1 + 3m)/2 & \text{for } m \text{ odd}
\end{cases}
\]

can be checked 5). Finally, the relation

\[
A_{n,m} = C_{2n,m} + n - n^2
\]

has a simple physical interpretation. A general \( n \)-port can be realized as a lossless non-reciprocal \( 2n \)-port of the same degree closed on \( n \) resistances, but the
realization is not unique since there exist lossless $2n$-ports of degree zero which transform $n$ separate resistances into $n$ separate resistances. In (5) the terms $C_{2n,m} + n$ are the number of parameters of an arbitrary realization, while $n^2$ is the number of degrees of freedom resulting from the non-uniqueness of the realization, as will now be shown. An arbitrary lossless $2n$-port of degree zero contains $C_{2n,0} = n(2n - 1)$ parameters, whereas the condition that a termination on $n$ resistances produces $n$ separate resistances at the input imposes $n(n - 1)$ relations (all non-diagonal elements of the input impedance matrix must vanish). The remaining number of degrees of freedom is $n(2n - 1) - n(n - 1) = n^2$.

The expressions (1), (2) and (3) are linear in the degree $m$, the coefficients of $m$ being $2n, n + 1$ and $n$, respectively. The values of the terms independent of $n$ are well known since they correspond to $m = 0$. The coefficients of $m$ can be obtained by the following simple method holding in all three cases. A rational impedance matrix can be expanded into simple fractions; moreover, all poles are generally distinct (multiple poles are obtainable by confluence and this does not increase the number of parameters). The degree of each simple fraction is the rank of its residue matrix and the total degree is the sum of the degrees of the terms 6). The number $k$ of real parameters of a constant matrix of order $n$ and rank $r$ has been computed 2) and is easy to derive by enumerating the relations imposed by the linear dependence and other restrictions. This number is $r(2n - r)$ for a general real matrix, and also for a Hermitian matrix; it is $r(2n - r + 1)/2$ for a real symmetric matrix. A simple fraction $A/(p - p_0)$ with real $p_0$, and thus real $A$, contains $k + 1$ parameters ($p_0$ is the additional parameter), thus $(k + 1)/r$ parameters per unit of degree; if $p_0$ and $A$ are complex, the number of real parameters is doubled in one simple fraction, but the conjugate fraction contains no new parameters so that the same number is obtained per unit of degree.

In the general case, the above computation holds with $k = r(2n - r)$ and the ratio $(k + 1)/r$ is maximum for $r = 1$, taking the announced value $2n$.

In the reciprocal case, each residue matrix is symmetrical and one must use $k = r(2n - r + 1)/2$; the ratio $(k + 1)/r$ is again maximum for $r = 1$ and takes the value $n + 1$.

In the lossless non-reciprocal case, the poles are at $p = 0$ or $p = \infty$ with real symmetric residue matrices, or occur in imaginary pairs with conjugate Hermitian residue matrices. For poles at 0 or $\infty$, one has $k = r(2n - r + 1)/2$ and the ratio is $k/r = (2n - r + 1)/2$; this is maximum for $r = 1$ and takes the value $n$. For an imaginary pole, one has $k = r(2n - r)$ and the ratio to be considered is $(k + 1)/2r$, in order to account for the conjugate fraction; this is also maximum for $r = 1$ and takes the same value $n$.

The lossness reciprocal case is more difficult, since $n$-ports having neither impedance nor admittance matrices have to be considered (this is obvious in the case $m = 0$). The proof consists of two parts: it is first shown than an $n$-port
having the number of independent parameters given by (4) can actually be constructed; it is then shown that an n-port having a larger number of independent parameters does not exist, by proving that every n-port has a realization using that number of elements. The construction procedure of the first part is different depending on the value of $n$ relative to $m$: for $n \leq m$, it will appear that impedance matrices exist which have the number of parameters given by (4); on the contrary, for $n > m$ (and in particular for $m = 0$), the construction must use additional ideal transformers.

Let us start by assuming the existence of an impedance matrix. Its poles are at $p = 0$ or $p = \infty$, or occur in imaginary pairs, but have always real symmetric residue matrices. For poles at 0 or $\infty$, the number of parameters per unit of degree is $(2n - r + 1)/2$ as in the last case, and the maximum value $n$ occurs for $r = 1$. For an imaginary pole, one still has $k = r (2n - r + 1)/2$ but the ratio to be considered is $(k + 1)/2r$; this is maximum for $r = 1$ and takes the much smaller value $(n + 1)/2$. One thus has to favour poles at zero and infinity up to some multiplicity, by adopting residue matrices of ranks $r_0$ and $r_\infty$ ($> 1$). The remaining terms, accounting for a total degree $m - r_0 - r_\infty$, must be produced by pairs of imaginary poles with residue matrices of rank one. The total number of parameters is then

$$r_0(2n - r_0 + 1)/2 + r_\infty(2n - r_\infty + 1)/2 + (m - r_0 - r_\infty)(n + 1)/2,$$

and one must choose $r_0$ and $r_\infty$ so as to maximize (6), with the restriction that

$$m - r_0 - r_\infty \geq 0$$

must be even. Since (6) is symmetric with respect to $r_0$ and $r_\infty$, the maximum occurs for $r_0 = r_\infty$ and (6) reduces to $m(n + 1)/2 + r_0(n - r_0)$, which is maximum for $r_0 = r_\infty = n/2$. These results only hold when $m$ and $n$ are both even, and (7) requires $m \geq n$. For other cases one must adopt integers in the neighbourhood of $n/2$, and a simple discussion of the value of (6) leads to the following optimal results

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>$r_\infty$</th>
<th>$D_{nm} = m(n+1)/2 +$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even $m$, even $n$</td>
<td>$n/2$</td>
<td>$n/2$</td>
</tr>
<tr>
<td>even $m$, odd $n$</td>
<td>$(n±1)/2$</td>
<td>$(n±1)/2$</td>
</tr>
<tr>
<td>odd $m$, even $n$</td>
<td>$n/2$, or $n/2±1$</td>
<td>$n/2±1$, or $n/2$</td>
</tr>
<tr>
<td>odd $m$, odd $n$</td>
<td>$(n±1)/2$</td>
<td>$(n±1)/2$</td>
</tr>
</tbody>
</table>

The condition (7) becomes $m \geq n$ if $m + n$ is even and $m \geq n + 1$ if $m + n$ is odd. The last column of the table coincides with (4), so that a realization with that number of parameters is established for $n \leq m$. 

In order to obtain a realization with $D_{nm}$ elements for $n > m$, we start from an $m$-port of degree $m$, thus having $D_{mm}$ independent parameters. We then form a $q$-port by using the $m$ original ports and defining the voltages at the $q - m$ additional ports as linear combinations of the voltages at the $m$ original ports; this adds $m(q - m)$ parameters (transformer ratios). From the $q$-port we finally form an $n$-port by using the $q$ ports and defining the currents at the $n - q$ additional ports as linear combinations of the currents at the $q$ existing ports; this adds $q(n - q)$ parameters. By this means we have formed an $n$-port of degree $m$ having

$$D_{mm} + m(q - m) + q(n - q)$$  \hspace{1cm} (8)

parameters (for the parameters at the various stages are clearly independent of each other), and this holds true for arbitrary integers $q$ such that $m \leq q \leq n$. The part

$$\alpha = q(m + n - q)$$  \hspace{1cm} (9)

of (8), which depends on $q$, is maximum when $q$ is the integer nearest to $(n + m)/2$ and the resulting maximum is

$$\alpha_{\text{max}} = (n + m)^2/4 - [1 - (-1)^{n+m}]/8.$$  \hspace{1cm} (10)

Finally, the maximum of (8) is $D_{mm} + \alpha_{\text{max}} - m^2$. The replacement of $\alpha_{\text{max}}$ by (10) and of $D_{mm}$ by the particular value resulting from (4) yields the general value (4) of $D_{nm}$.

To prove the second part, we consider the realization of the $n$-port of degree $m$ by means of a transformer $(n + m)$-port closed on $m$ separate reactive elements: $\lambda$ inductances and $\gamma = m - \lambda$ capacitances. The realization is not unique, for transformer $2\lambda$-ports exist that change $\lambda$ separate inductances into $\lambda$ separate inductances, and similarly for capacitances. The number of degrees of freedom of such a transformation is the same as for an orthogonal matrix of order $\lambda$, thus $\lambda(\lambda + 1)/2$. The number of ratios of a transformer $(n + m)$-port is given by (9), where $q$ is some integer. One may, however, make use of the non-uniqueness by imposing

$$\beta = [\lambda(\lambda + 1) + \gamma(\gamma + 1)]/2$$  \hspace{1cm} (11)

arbitrary conditions (for instance by making certain ratios reduce to 0 or 1) and this reduces the number of ratios of the transformer $(n + m)$-port accordingly. The total number of elements in the realization is then

$$\alpha - \beta + m,$$  \hspace{1cm} (12)

the last term representing the reactive elements. The maximum of (12) is reached for the value of $q$ which maximizes (9) and the value of $\lambda$ which minimizes (11),
with the constraint $\gamma = m - \lambda$. The maximum with respect to $q$ is the one discussed in the last paragraph and replaces $a$ by $a_{\text{max}}$. The minimum of (11) with respect to $\lambda$ occurs when $\lambda$ is the integer nearest to $m/2$, and the resulting value of (11) is

$$\beta_{\text{min}} = (m^2 + 2m)/4 + [1 - (-1)^m]/8.$$  \hspace{2cm} (13)

With the extremal values (10) and (13), (12) reduces to (4).

The author is grateful to Prof. B. D. H. Tellegen for stimulating discussion and advice.

MBLE Research Laboratory Brussels, May 1963

REFERENCES

6) B. McMillan, Introduction to formal realizability theory, Bell Syst. tech. J. 31, 217-279, 541-600, 1952; see p. 543, theor. 2.15.