RIGOROUS CALCULATION OF THE ELECTROMAGNETIC FIELD OF WAVE BEAM

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Abstract

The rigorous calculation of the electromagnetic field of wave beams is treated by using a combination of a cylindrical- and a rectangular-coordinate system. Cylindrical coordinates are used for the space variables while the electromagnetic field is uniquely defined by the rectangular components $E_z(r, \theta, z)$ and $E_y(r, \theta, z)$ in accordance with the optical representation. With this procedure the components of the electromagnetic field are obtained from the boundary conditions without any approximation, and the error due to the paraxial approximations, used by Goubau, is calculated. Finally a review of the characteristics of the beam waveguide is given.

1. Introduction

Wave beams are applied in a large wave range, from optics till microwave techniques. The use of wave beams in radar, microwave links, etc., involves some optical devices such as: mirrors, gratings and lenses, used by Goubau for guiding microwaves. A system of the latter kind, called a beam waveguide, is shown in fig. 1. The lenses, correcting the diffraction of the beam, concentrate the energy in the neighbourhood of the propagation axis.

In this paper the electromagnetic field of wave beams will be investigated by using cylindrical coordinates (see fig. 2), this procedure being specially suited to beams possessing rotational symmetry. The separation of the scalar wave equation then leads to a Bessel equation. In a homogeneous, isotropic domain...

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Fig. 1. Beam waveguide.

Fig. 2. Coordinate system.
The time-harmonic electric field can therefore be represented by linear combinations of the elementary wave functions \(^2\)

\[
\Psi_{mk} = e^{im\theta} J_m (\sqrt{k^2 - h^2} r) e^{\pm i\omega t},
\]

(1)

where \(h\) is the propagation constant, \(k = \omega \sqrt{\varepsilon \mu}\) the wave number which is real for the lossless domain, assumed henceforth, and \(m\) an integer. The Bessel function of the first kind only can be used, the other functions becoming finite at \(r = 0\). The half space \(z \geq 0\) is considered, while the sources of the field are assumed to be in the other half space \(z < 0\). The equation (1) can then also be represented by

\[
\Psi_{mk} = e^{im\theta} J_m (\lambda r) e^{-\sqrt{\lambda^2 - k^2} z - i\omega t},
\]

(2)

here \(\sqrt{k^2 - h^2} = \lambda\) and \(\sqrt{\lambda^2 - k^2} > 0\) if \(\lambda > k\); for \(\lambda < k\) we may substitute \(\sqrt{\lambda^2 - k^2} = -i \sqrt{k^2 - \lambda^2}\), so as to obtain waves propagating to infinity in the half space under consideration.

To find the complete solution of the wave equation for fixed \(k\) we must multiply each elementary wave function by a weight function \(\lambda g_m(\lambda)\) (written in this form for mathematical convenience only) and integrate with respect to \(\lambda\). After a summation over \(m\) this results in

\[
\Psi = e^{-i\omega t} \sum_{m = -\infty}^{\infty} e^{im\theta} \int g_m(\lambda) J_m(\lambda r) e^{-\sqrt{\lambda^2 - k^2} z} \lambda d\lambda.
\]

(3)

The axial field components \(E_z\) and \(H_z\) can be calculated from the boundary conditions at \(z = 0\) by using the relation (3) for each of them. The transverse components are the most interesting ones, because generally they are measured and are greater than the axial components. The components \(E_\theta, E_r, H_\theta\) and \(H_r\) are found from \(E_z\) and \(H_z\) by applying the Maxwell equations. As the expressions become rather complicated, Goubau used an approximation to simplify them. However, fairly simple relations can also be obtained without any approximation at all, by means of the following method.

In waveguide theory the e.m. fields are given by a linear combination of a transverse electric field \((E_z = 0)\) and a transverse magnetic field \((H_z = 0)\). In optics, however, the e.m. fields are often given by a linear combination of the electric fields in the two polarization directions, so that, in terms of waveguide theory, either \(E_z = 0\) or \(E_y = 0\). The optical representation will be used here. In view of the assumed homogeneity and isotropy of space the time-harmonic fields can be determined, with the aid of (3), from the boundary values at \(z = 0\) of the transverse components \(E_x\) and \(E_y\), see appendix I.

The expressions for \(E_x\) and \(E_y\), calculated directly from the wave equation, are simpler than the expressions for \(E_r\) and \(E_\theta\) determined via the axial components \(E_z\) and \(H_z\).
2. Calculation of the e.m. field

The complete electromagnetic field will be derived here from a linear combination of the two field components $E_x(r, \theta, z)$ and $E_y(r, \theta, z)$. The calculation of either $E_x$ or $E_y$ from its boundary value at $z = 0$ is analogous; on the expression for $E_x$ is derived below.

The problem is to determine $E_x(r, \theta, z)$ for all positive values of $z$ from its boundary value at $z = 0$, given by the distribution $E_x(r, \theta)$, which is assumed to be bounded, single-valued function of its variables and piecewise continuous together with its first derivatives. Therefore $E_x(r, \theta)$ can be expanded in a Fourier series:

$$E_x(r, \theta) = \sum_{m = -\infty}^{\infty} E_{xm}(r) e^{im\theta},$$

where

$$E_{xm}(r) = \frac{1}{2\pi} \int_{0}^{2\pi} E_x(r, \theta) e^{-im\theta} d\theta.$$

For $z = 0$ the time-independent part of relation (3) becomes

$$E_x(r, \theta) = \sum_{m = -\infty}^{\infty} e^{im\theta} \int_{0}^{\infty} e_{xm}(\lambda) J_m(\lambda r) \lambda d\lambda,$$

where

$$e_{xm}(\lambda) = g_m(\lambda).$$

The function $e_{xm}(\lambda)$ can be found from (5) for a given distribution $E_x(r, \theta)$. The following method for evaluating (5) can be used for Bessel functions of orders $\geq 0$ only. However, Bessel functions of negative orders can be transformed into those of positive orders, because $m$ is an integer and consequently

$$J_m(\lambda r) = (-1)^m J_{-m}(\lambda r).$$

Hence by the substitutions

$$e_{xm} = e^{+}_{xm}, \quad E_{xm} = E^{+}_{xm} \quad (m \geq 0)$$

and

$$(-1)^m e_{xm} = e^{-}_{xm}, \quad E_{xm} = E^{-}_{xm} \quad (m < 0),$$

the relations (4) and (5) can be transformed respectively into

$$E_x(r, \theta) = \sum_{\pm} \sum_{m = 0}^{\infty} E^{\pm}_{xm}(r) e^{\pm im\theta}$$

and

$$E_x(r, \theta) = \sum_{\pm} \sum_{m = 0}^{\infty} e^{\pm im\theta} \int_{0}^{\infty} e^{\pm}_{xm}(\lambda) J_m(\lambda r) \lambda d\lambda.$$
In view of (6a) and (6b) we have

\[ E_{\pm zm}(r) = \int_0^\infty e^{\pm zm(\lambda)} J_m(\lambda r) \lambda d\lambda \quad (m \geq 0). \] (7)

The Fourier-Bessel transform of (7) leads to

\[ e^{\pm zm}(\lambda) = \int_0^\infty E_{\pm zm}(r) J_m(\lambda r) r \, dr \quad (m \geq 0). \] (8)

In order to evaluate (8) we expand \( E_{\pm zm}(r) \) into orthogonal functions \( R_{nm}(x^2) \) connected with the Laguerre polynomials \( L_{nm}(x^2) \) \( (m \geq 0 \text{ is now required}) \). The functions \( R_{nm}(x^2) \) are given explicitly by

\[ R_{nm}(x^2) = x^m e^{-\frac{1}{2} x^2} L_n^m(x^2) = x^m e^{-\frac{1}{2} x^2} \sum_{k=0}^{\infty} \frac{(m+n)!}{(n-k)! (m+k)! k!} (-x^2)^k. \] (9)

In appendix II some properties of these functions are discussed. The expansion in question reads

\[ E_{\pm zm}(r) = \sum_{n=0}^{\infty} a_{\pm mn} R_{nm}(\frac{r^2}{r_0^2}), \] (10)

where

\[ a_{\pm mn} = \frac{n!}{(m+n)!} \int_0^\infty E_{\pm zm}(r) R_{nm}(\frac{r^2}{r_0^2}) \, d\frac{r^2}{r_0^2}. \] (11)

The factor in front of the integral is a constant of normalization while \( r_0 \) is an arbitrary constant, which value can still be chosen conveniently. A suitable value of \( r_0 \) corresponds often to the radius of the wave beam. The equation (8) can now be expanded as follows:

\[ e^{\pm zm}(\lambda) = \sum_{n=0}^{\infty} a_{\pm mn} \int_0^\infty R_{nm}(\frac{r^2}{r_0^2}) J_m(\lambda r) \, r \, dr. \] (12)

The integration then yields (see appendix II)

\[ e^{\pm zm}(\lambda) = \sum_{n=0}^{\infty} (-1)^n a_{\pm mn} r_0^2 R_{nm}(r_0^2 \lambda^2). \] (13)

The coefficients \( a_{\pm mn} \) of (12) and (13) can be derived from the boundary conditions by using the relations (4) and (11).
Next, the electromagnetic field for \( z \neq 0 \) follows from (3). The wave propagating in the positive \( z \)-direction thus proves to result into

\[
E_z(r, \theta, z) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{\pm \frac{1}{k} \lambda \alpha} (-1)^n a_{mn} \int_0^\infty R_n^m(r_0 \lambda \alpha) J_m(\alpha r) e^{-\frac{\alpha}{\lambda} z - k^2 z} \lambda d\lambda,
\]

remembering that

\[
g_m(\lambda) = e_{xm}(\lambda) .
\]

Each rectangular component can be expanded into such a series. The individual terms are (erroneously) called modes. The order of a special one, characterized by the values of \( m \) and \( n \), will be indicated henceforth as \((m, n)\). The simplest mode is thus characterized by the order \((0, 0)\). Goubau concluded that a beam with this mode must have the lowest losses in the beam waveguide.

3. The electric field of the beam with the mode of order \((0, 0)\)

The electric field can be calculated from the relation (14) which reduces for \( m = n = 0 \) to

\[
E_{z00}(r, z) = r_0^2 a_{00} \int_0^\infty e^{-\frac{1}{k^2} r_0^2 \alpha} J_0(\alpha r) e^{-\frac{\alpha^2}{\lambda} z - k^2 z} \lambda d\lambda .
\]

At \( z = 0 \), (15) can be evaluated rigorously, yielding

\[
E_{z00} (r, 0) = a_{00} e^{-r^2/2 r_0^2} .
\]

The electric field of the mode of order \((0, 0)\) thus proves to have a Gaussian distribution over the plane \( z = 0 \). The expression (15) for \( z \neq 0 \) then determines the radiation field of a Gaussian aperture distribution.

In order to evaluate (15) another expansion in terms of Laguerre functions will be used, viz.

\[
e^{-\frac{1}{k^2} r_0^2 \alpha} \frac{1}{\lambda^2 - k^2} z = \sum_{i=0}^{\infty} \beta_{0i}(z) R_0^i \left( \frac{\lambda^2}{\lambda_0^2} \right) ,
\]

where

\[
\beta_{0i}(z) = \int_0^\infty e^{-\frac{1}{k^2} r_0^2 \alpha} \frac{1}{\lambda^2 - k^2} z R_0^i \left( \frac{\lambda^2}{\lambda_0^2} \right) \frac{\lambda^2}{\lambda_0^2} .
\]

The substitution of (17) into (15) yields (see appendix II)

\[
E_{z00} (r, z) = \lambda_0^2 r_0^2 a_{00} \sum_{i=0}^{\infty} (-1)^i \beta_{0i}(z) R_0^i \left( \lambda_0^2 r_0^2 \right) .
\]
The Gaussian distribution thus appears to be deformed, higher-order modes being generated by the propagation. In optics the deformations are observed as fringes, and in the antenna theory as side lobes.

The yet arbitrary constant $\lambda_0$ can be chosen such that the series will converge most rapidly. For the range $kr_0 \gg 1$, a suitable value of $\lambda_0$ can be obtained as follows. The integrand of (15) only differs significantly from zero if $r_0 \lambda \ll 1$, which then implies $\lambda \ll k$ for the range in question. Hence we may apply the approximation on which Goubau has based his treatment from the beginning:

$$\sqrt{\lambda^2 - k^2} \sim -ik \left(1 - \frac{1}{2} \frac{\lambda^2}{k^2}\right),$$

the sign on the right-hand side being in accordance with the remarks following expression (2). The corresponding evaluation of (15) results in

$$E_{200}(r, z) \sim \frac{a_{00} r_0^2}{r_0^2 + iz/k} \exp \left[-\frac{r^2}{2(r_0^2 + iz/k)}\right] e^{ikz}.$$  \hspace{1cm} (19)

Hence, in the range under consideration ($r_0k \gg 1$), the Gaussian distribution is not deformed by the propagation in a first approximation, no higher-order modes being formed.

According to (18) the exact contribution of the zero mode after propagation over a distance $z$, will be

$$E_{200,0}(r, z) = a_{00} r_0^2 \lambda_0^2 \exp \left[-\frac{i}{2} \frac{\lambda_0^2 r_0^2}{k} \beta_{00}(z)\right].$$ \hspace{1cm} (20)

Comparing (19) and (20) we infer a complete agreement with respect to the dependence, if, for the range in question, $\lambda_0$ is determined by

$$\lambda_0^2 = \frac{1}{r_0^2 + iz/k}.$$

The function $\beta_{00}(z)$, to be derived from (17) and (9), reads

$$\beta_{00}(z) = \left(r_0^2 + \frac{iz}{k}\right) \int_0^\infty e^{-\left(r_0^2 + \frac{iz}{k}\right)x^2 - q x} (x - ik) \, dx.$$ \hspace{1cm} (21)

We next substitute

$$\sqrt{\lambda^2 - k^2} = x - ik,$$

in accordance with the remarks concerning (2); the integral (21) then reduces to

$$\beta_{00}(z) = 2 e^{ikz} \left(r_0^2 + \frac{iz}{k}\right) \int_0^\infty e^{-p x^2 - q x} (x - ik) \, dx.$$ \hspace{1cm} (22)

where

$$p = r_0^2 + \frac{iz}{2k},$$
and
\[ q = 2(z - i r_0^3 k). \]

A further linear shift of the integration variable \( x \) permits a reduction to the error function \( \Phi \) according to
\[
\beta_{00}(z) = e^{ikz} \left( r_0^3 + \frac{iz}{k} \right) \left[ \frac{1}{p} + \sqrt{\frac{\pi}{p}} e^{q^2/4p} \left\{ 1 - \Phi \left( \frac{q}{2\sqrt{p}} \right) \right\} - ik - \frac{q}{2p} \right]. \tag{23}
\]

The error function may be approximated by the first two terms of its asymptotic expansion in the case
\[
\frac{2p}{q^2} \ll 1.
\]

Figure 3 shows the representation of the correction factor due to cutting its expansion after its second term. The approximation involves
\[
\Phi \left( \frac{q}{2\sqrt{p}} \right) \approx 1 - \frac{2}{q} \sqrt{\frac{p}{\pi}} e^{q^2/4p} \left( 1 - \frac{2p}{q^2} \right). \tag{24}
\]

By substituting (24) into (23) we obtain
\[
\beta_{00}(z) \approx 2 e^{ikz} \left( r_0^3 + \frac{iz}{k} \right) \left( - \frac{ik}{q} + \frac{q + 2 ikp}{q^3} \right) \tag{25}
\]
or
\[
\beta_{00}(z) \approx e^{ikz} \left( 1 - \frac{z}{4 ik (z - ir_0^3 k)^2} \right). \tag{26}
\]

Application of (26) transforms (20) into
\[
E_{00,0}(r, z) \approx \frac{a_{00}}{r_0^3 + iz/k} \exp \left[ - \frac{r^2}{2(r_0^3 + iz/k)} \right] e^{ikz} \left( 1 - \frac{z}{4 ik (z - ir_0^3 k)^2} \right). \tag{27}
\]

Comparing (19) with (27) we obtain the second-order correction factor
\[
U = 1 - \frac{z}{4 ik (z - ir_0^3 k)^2},
\]
which determines the correction for both the amplitude and the phase. The

Fig. 3. Representation of the correction factor due to truncating the asymptotic series for the error function \( r_0 k \gg 1 \).
focus of \( U \) in the complex plane proves to be a circle, as illustrated in fig. 4. For the amplitude the correction effect is maximum for \( z = r_0^2k \), where \( U \) passes through its minimal value

\[
1 - \frac{1}{8r_0^2k^2};
\]

at this very value the fraction of the energy concentrated in the higher modes is also maximum. For the phase the maximum correction effect is given by

\[
\alpha = \arcsin \left( \frac{1}{16r_0^2k^2 - 1} \right).
\]

For a beam having at \( z = 0 \) a diameter equal to the wavelength the factor \( r_0k \) is about 3, and the maximum errors introduced by the factor in question are about 1·3% for the amplitude and 0·007 radians for the phase. Hence it may be concluded that the relation (19), which disregards the correction factor, may be used in all cases in which the smallest diameter of the beam exceeds the wavelength.

4. Some characteristics of the beam with mode of order \((0, 0)\)

The beam having a field distribution of the mode of order \((0, 0)\) has been described by Goubau \(^3\). A review of its most important features will be given below. The relations to be discussed will be derived from (19) and therefore can be used in the range \( kr \ll 1 \) only. According to (19) we start from a Gaussian distribution over the plane \( z = 0 \), which constitutes a plane wave front with respect to the phase.

The energy flux through the circular region \( \rho < r \) in a plane \( z = \) constant can be calculated from the relation

\[
\int_0^\infty E_{x00} (\rho, z) E^{*}_{x00} (\rho, z) 2\pi \rho \, d\rho = \pi a_{00}^2 r_0^2 \left[ 1 - \exp \left( -\frac{r^2r_0^2k^2}{r_0^4k^2 + z^2} \right) \right]. \tag{28}
\]

The tubes of energy flux, defined by the property that a constant amount of
energy passes through their varying cross-section with radius \( r(z) \), are determined by the equation

\[
C = \frac{r^2 r_0^2 k^2}{r_0^4 k^2 + z^2},
\]

in which the constant \( C \) differs for each tube. The local diameter \( r_b \) of the beam can be defined as that of a special tube, for which \( C = 1 \); see fig. 5. We thus find

\[
r_b^2 = \frac{z^2}{r_0^4 k^2} + r_0^2. \tag{29}
\]

![Diagram](image)

Fig. 5. Diameter of the beam with the mode of order \((0, 0)\).

All tubes \( C = \text{constant} \) prove to be hyperboloids of revolution. The tube \( C = 1 \) has an asymptotic cone with an angular width given by

\[
\tan \alpha = \frac{1}{r_0 k} \tag{30}
\]

This relation can be used down to \( r_0 k = 3 \), which restricts the flare angle to a value below 18°.

The approximation (19) fixes the dependence of the phase angle \( \phi \) on \( z \) and \( r \). We find

\[
\phi = kz - \arctan \left( \frac{z}{k r_0^2} \right) + \frac{r^2 k z}{2 \left( r_0^4 k^2 + z^2 \right)} \tag{31}
\]

The term depending on \( r \) involves a deviation of the phase fronts from plane surfaces. The corresponding disturbance depends on the contribution

\[
\phi_r = \frac{r^2 k z}{2 \left( r_0^4 k^2 + z^2 \right)} \tag{32}
\]

A lens compensating this phase distortion can be used in order to recover the plane phase front. A lens situated at \( z = a \) should then produce the phase correction.
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\[ (\phi)_{\alpha} = \frac{r^2 k a}{2 (r_{0a}^4 k^2 + a^2)} = \frac{kr^2}{2 F}. \]  

(33)

This can be achieved by a lens whose profile is parabolic with respect to its effective optical thickness as a function of the distance to the axis. It can be proved with the aid of paraxial approximations that \( F \) constitutes the focal distance of the lens. The relation between the distance \( a \), fixing the position of the lens, and the corresponding focal distance \( F \) reads, in view of (33),

\[ F = a \left(1 + \frac{r_{0a}^4 k^2}{a^2}\right). \]  

(34)

For the limiting wavelength \( \lambda \to 0 \) we have \( k \to \infty \) and thus \( F \to \infty \). For zero wavelength the diffraction vanishes and the required focal distance is thus increased to infinity.

We now consider a new situation (see fig. 6) with a lens at \( z = 0 \), and an undistorted Gaussian distribution occurring at \( z = -a \) instead of \( z = 0 \). We assumed the latter to be imaged by this lens in the plane \( z = b \). The question arises how to choose the focal distance \( F \) and the diameter of the lens in order to realize this situation. The total phase correction of the lens, compensating the distortions produced along the sections \( -a < z < 0 \) and \( 0 < z < b \), must then be

\[ \frac{ak r^2}{2 (r_{0a}^4 k^2 + a^2)} + \frac{b k r^2}{2 (r_{0b}^4 k^2 + b^2)} = \frac{kr^2}{2 F}, \]  

(35)

so that

\[ \frac{1}{a + r_{0a}^4 k^2/a} + \frac{1}{b + r_{0b}^4 k^2/b} = \frac{1}{F}. \]  

(36)

The focal distance can be calculated from (36), \( a \) and \( b \) being prescribed. Further, the diameter of the lens should exceed the beam diameter \( 2r_b \) at \( z = 0 \). The latter is given, in view of (29), by

\[ r_b^2 = \frac{a^2}{r_{0a}^2 k^2} + r_{0a}^2 = \frac{b^2}{r_{0b}^2 k^2} + r_{0b}^2. \]  

(37)

Fig. 6. Imaging in \( z = b \) of an undistorted Gaussian field distribution occurring in \( z = -a \).
It can be proved that in the range $z > 0$, the highest intensity of the field occurs in the plane $z = b$.

The case $a = 0$ corresponds to an incident beam with an undistorted Gaussian field distribution, thus constituting a plane phase front at the lens itself. In the image space the plane having the highest intensity is then given by

$$z = b = \frac{F}{1 + \frac{F^2}{k^2} r_b^4},$$

(38)

see (36) and (37).

For the limiting wavelength $\lambda_0 \to 0$ we have $b = F$. In other words, for zero wavelength, i.e. for vanishing diffraction, the highest intensity occurs in the focus, but in cases of noticeable diffraction, it occurs closer to the lens.

In the beam waveguide, composed of a set of lenses, the image formed by the preceding lens is the object for the following one (see fig. 1). The diffraction is minimum for a beam with a field distribution of the mode of order $(0, 0)$. The diffraction losses are caused by the finiteness of the lens diameters, and therefore the radius $r_b$ of the beam at the lenses should be as small as possible. For a given distance $a$ or $b$ the diameter $r_b$ is minimum, in view of (37), if

$$r_{0a}^2 = \frac{a}{k}, \quad r_{0b}^2 = \frac{b}{k},$$

respectively. Under these optimum conditions we have

$$a = b = r_{0a}^2 k = r_{0b}^2 k = F, \quad \left\{ \begin{array}{l} r_{0a}^2 = 2 r_{0a}^2 = 2 r_{0b}^2. \end{array} \right.$$ (39)

This implies equal values of the two contributions to the focal distance, as represented by (36); each lens is then situated at the transition from the Fresnel to the Fraunhofer zone of the image formed by the preceding lens. In fact, for structures with characteristic dimensions $r_0$ this transition occurs at a distance $a$ of the order of $k r_0^3$ or $r_0^2 / \lambda$, that is in the situation where both terms of (37) become equal.

From the correction factor $U$ and (39) it follows that the fraction of the energy concentrated in the higher modes is maximum in the focal planes.

The beam waveguide is characterized by the relations (39); the lenses must be placed at a distance $2F$ from each other in order to obtain minimum diffraction losses.

5. Beams with modes of higher order

The electromagnetic fields of beams having a field distribution with modes of higher order will now be calculated, assuming $k r_0 \gg 1$; the approximation

$$\sqrt{\lambda^2 - k^2} \sim - i k \left(1 - \frac{1}{2} \frac{\lambda^2}{k^2}\right)$$
then holds. Equation (14) can thus be written in a first approximation as

\[ E_x(r, \theta, z) \sim e^{ikz} \sum_{\pm m=0}^{\infty} \sum_{n=0}^{\infty} e^{\pm im\theta} \left(-1\right)^m a_{mn} \int_0^{\infty} \exp\left(-\frac{izp^2}{2kr_0^2}\right) R_n^m(p^2) J_m\left(\frac{r}{r_0}\right) p \, dp, \]

where \( p = r_0 \lambda \).

The evaluation (see appendix II) yields

\[ E_x(r, \theta, z) \sim \frac{\exp\left[ikz\left(1 + \frac{r^2}{2(k^2r_0^4 + z^2)}\right)\right]}{1 + iz/k r_0^2} \sum_{\pm m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left(\frac{1 - iz/k r_0^2 + i - k^2r_0^2 + z^2}{1 + iz/k r_0^2}\right)^{\pm im\theta}. \]

Obviously the curvature of the modified phase planes only depends on the exponential factor in front of the summation; the distortions of the phase planes thus appear to be identical for all modes. Further the distribution of the intensity over a plane \( z = \text{constant} \) depends for each mode, composed of the terms with the upper and lower signs, on

\[ \left\{ R_n^m\left(\frac{r^2}{r_0^2 (1 + z^2/k^2r_0^4)}\right)\right\}^2 \left\{1 + \cos (2m\theta + \phi)\right\}; \]

therefore the pattern is enlarged at increasing values of \( z \), and the factor of enlargement

\[ 1 + \frac{z^2}{k^2r_0^4} \]

is equal for all modes.

The phase velocities of each mode, however, prove to be dependent on \( m \) and \( n \). We first consider the simplest configuration which corresponds to \( n = 0 \). Each individual mode of order \((m, 0)\) then leads to an intensity pattern in which the field is concentrated in \( 2m \) local dots on a single ring. The radial concentration here depends on the function

\[ R_0^m(x^2) = x^m e^{-\frac{1}{2} x^2}, \]

where

\[ x = \frac{r}{r_0 \sqrt{1 + z^2/k^2 r_0^4}}. \]
The location of the field maximum can be found from the relation

\[
\left[ R_0^m(x^2) \right]' = \left( m x^{m-1} - x^{m+1} \right) e^{-\frac{1}{2} x^2} = 0. \tag{43}
\]

The radius \( r_d \) of the ring on which the dots are situated can be calculated from (43):

\[
r_d = r_0 \sqrt{1 + \frac{z^2}{k^2 r_0^4} M}. \tag{44}
\]

As a practical application of this calculation we consider one of the pattern of a short gas laser, such as constructed by Haismia and De Lang \( ^4 \). Figure 7 shows a pattern of a beam generated in such a laser. This pattern consists of two rings, an exterior ring of 36 dots and an inner ring of 8 dots; so it is a superposition of two modes of the orders \((18,0)\) and \((4,0)\), respectively. The question arises whether the parameter \( r_0 \) has the same value for both rings. From (44) we conclude that, in that case, the diameters of the rings should have the ratio \(|18/4 = 2.1|\); however, the ratio measured from the picture proves to be 3.2. So these two modes have different values of \( r_0 \).

\[
\begin{array}{c|c|c|c}
  m & 0 & 1 & 2 \\
  \hline
  0 & \bullet & \bullet & \bullet \\
  1 & \circ & \circ & \circ \\
  2 & \circ & \circ & \circ \\
\end{array}
\]

Fig. 7. Mode pattern of a short gas laser.

Fig. 8. Schematic field pattern for beams with modes of order \( m \leq 2, n \leq 2 \).
Field patterns of the beams corresponding to a single mode of order \((m, n)\), up to \(m = n = 2\), are schematically shown in fig. 8. We will not dwell upon the conditions under which certain of these modes are actually excited in gas lasers.

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Appendix I

It must be proved that the time-harmonic electromagnetic field, proportional to \(e^{-i\omega t}\), can be uniquely determined from its components \(E_x(r, \theta, z)\) and \(E_y(r, \theta, z)\) in a homogeneous and isotropic medium. In particular it must be shown that the expansions for \(E_z, H_y, H_x, H_z\), to be derived from \(E_x\) and \(E_y\) in terms of wave-equation solutions, are the only ones satisfying the Maxwell equations while vanishing at \(z \to \infty\). The procedure will be worked out here for \(H_z\) and \(E_z\); the other components can be treated in a similar way.

From the Maxwell equations it follows that

\[
H_z = \frac{1}{i\omega \mu} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right). \tag{A 1}
\]

Therefore the component \(H_z\) is uniquely determined by \(E_y\) and \(E_x\). According to (3) the transverse electric components are given by

\[
E_p(r, \theta, z) = \sum_{m = -\infty}^{\infty} e^{im\theta} \int_0^{\infty} e_{p,m}(\lambda) J_m(\lambda r) e^{-\sqrt{\lambda^2 - k^2} z} \lambda \, d\lambda, \quad (p = x, y). \tag{A 2}
\]

Hence, in view of elementary properties of Bessel functions,

\[
\frac{\partial E_y}{\partial x} = \frac{1}{2} \sum_{m = -\infty}^{\infty} \int_0^{\infty} \lambda e_{y,m}(\lambda) \left[ -J_{m+1}(\lambda r) e^{i(m+1)\theta} + 
\right.
\]

\[
+ J_{m-1}(\lambda r) e^{i(m-1)\theta} \bigg] e^{-\sqrt{\lambda^2 - k^2} z} \lambda \, d\lambda. \tag{A 3}
\]

Assembling the terms with identical exponents we find

\[
\frac{\partial E_y}{\partial x} = \frac{1}{2} \sum_{m = -\infty}^{\infty} e^{im\theta} \int_0^{\infty} \lambda \left\{ -e_{y,m-1}(\lambda) + e_{y,m+1}(\lambda) \right\} J_m(\lambda r) e^{-\sqrt{\lambda^2 - k^2} z} \lambda \, d\lambda. \tag{A 4}
\]
In a similar way we derive

\[ \frac{\partial E_x}{\partial y} = \frac{1}{2} \sum_{m=-\infty}^{\infty} e^{im\theta} \int_{0}^{\infty} i\lambda \{ e_{x,m-1}(\lambda) + e_{x,m+1}(\lambda) \} J_m(\lambda r) e^{-\sqrt{\lambda^2 - k^2} r} \lambda d\lambda. \quad (A5) \]

By substituting (A 4) and (A 5) into (A 1), it follows that

\[ H_x(r, \theta, z) = \sum_{m=-\infty}^{\infty} e^{im\theta} \int_{0}^{\infty} h_{z,m}(\lambda) J_m(\lambda r) e^{-\sqrt{\lambda^2 - k^2} r} \lambda d\lambda, \quad (A6) \]

where

\[ h_{z,m}(\lambda) = \frac{\lambda}{2i\omega \mu} \left\{ -e_{y,m-1}(\lambda) + e_{y,m+1}(\lambda) - i \epsilon_{x,m-1}(\lambda) - i \epsilon_{x,m+1}(\lambda) \right\}. \]

We have thus derived the expression of \( H_x(r, \theta, z) \), following uniquely from the given distributions (A 2) for \( E_x(r, \theta, z) \) and \( E_y(r, \theta, z) \).

The components \( E_x(r, \theta, z) \) can be calculated from

\[ \frac{\partial E_x}{\partial z} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y}, \quad (A7) \]

yielding after integration

\[ E_x(r, \theta, z) = E_x^0(r, \theta, z) + \zeta(r, \theta), \quad (A8) \]

where \( E_x \) represents an expansion to be derived from \( E_x \) and \( E_y \) in the same way as \( H_x \) above; \( E_x^0 \) then appears as a new expansion but the integration involves a further contribution \( \zeta(r, \theta) \), independent of \( z \), which may be an arbitrary function of \( r \) and \( \theta \). However, the condition

\[ \lim_{z \to \infty} E_x(r, \theta, z) \to 0 \]

then implies \( \zeta(r, \theta) = 0 \); \( E_x^0 \) vanishes at \( z \to \infty \) in view of the exponent in its integral. Hence \( E_x \) is also uniquely determined by the two components \( E_x \) and \( E_y \).

Appendix II

The Laguerre polynomials have been discussed by Rainville 5). To give an impression of the Laguerre functions, we list those for \( n = 0, 1, 2 \):

\[ R_0^m(x^2) = x^m e^{-\frac{1}{2} x^2} \]

\[ R_1^m(x^2) = x^m e^{-\frac{1}{2} x^2} (1 + m - x^2) \]

\[ R_2^m(x^2) = x^m e^{-\frac{1}{2} x^2} \left\{ \frac{1}{2}(1 + m)(2 + m) - (2 + m) x^2 + \frac{1}{2} x^4 \right\} \]
Figures 9a and 9b show some of these functions; only the range $x' \geq 0$ is interesting in our case, because in our treatment the argument $x$ is proportional to the radial coordinate $r$.

The Laguerre functions are known to be orthogonal according to the relation

$$
\int_{0}^{\infty} R_{n}^{m}(x^2) R_{m}^{m}(x^2) \ dx^2 =
$$

$$
= \int_{0}^{\infty} x^{2m} e^{-x^2} L_{n}^{m}(x^2) L_{m}^{m}(x^2) \ dx^2 = \delta_{n}^{m} \frac{(m + n)!}{n!},
$$
\(\delta_{k}^{n}\) being the Kronecker symbol.

For the evaluation of the integrals we can use special cases of the following relation

\[
\int_{0}^{\infty} x^{v+\frac{1}{2}} e^{-\beta x^2} L_{n}^{v}(ax^2) J_{v}(xy) (xy)^{\frac{1}{2}} dx =
\]

\[
= 2^{-v-1} \beta^{-v-n-1} (\beta-a)^{n} y^{v+\frac{1}{2}} e^{-y^2/4\beta} L_{\frac{v-n-1}{2}}^{\left(\frac{\alpha y^2}{4\beta(a-\beta)}\right)},
\]

which has been proved \(^6\) to be even for complex \(v\). Applying this relation for \(\beta = \frac{1}{2}, a = 1\), we find

\[
\int_{0}^{\infty} R_{n}^{m}(x^2) J_{m}(xy) x dx = (-1)^{n} R_{n}^{m}(y^2),
\]

and for \(\beta = \frac{1}{2} + \frac{1}{2} i\xi, a = 1\):

\[
\int_{0}^{\infty} e^{-\frac{1}{2}i\xi x^2} R_{n}^{m}(x^2) J_{m}(xy) x dx =
\]

\[
= \frac{(-1)^{n}}{1 + i\xi} \frac{(1 - i\xi)^{n+m}}{(1 + i\xi)^{n-m}} R_{n}^{m} \left(\frac{y^2}{1 + \xi^2}\right).
\]

The latter relation enables us to evaluate the integral (14), provided we may use the approximation

\[
\sqrt{\lambda^2 - k^2} \sim -ik \left(1 - \frac{1}{2} \frac{\lambda^2}{k^2}\right).
\]

**REFERENCES**