LONGITUDINAL AND TRANSVERSE VOLTAGES
IN SUPERCONDUCTORS

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Abstract
Both in the intermediate state and in the mixed state of superconductors resistivity and Hall effect can be observed. This paper discusses a model for the transverse and longitudinal voltages across a sample as a function of the applied magnetic field. Both voltages are related to the motion of normal regions or vortices. The forces acting on a vortex are also discussed. The present calculations result in the introduction of a Magnus-type force, the magnitude of which depends, among other factors, on the purity of the material. The predicted Hall angle in the mixed state as a function of the applied magnetic field is somewhat larger than the extrapolated value of the normal state. A comparison of this theory with experimental results is made.

1. Introduction
In the intermediate state and in the mixed state of superconductors resistivity 1-5) and Hall effect 6) are observed. These phenomena are related to the motion of normal areas in type-I superconductors and to the motion of vortices in superconductors of the second kind.

In this paper we always consider a slab of a superconductor placed in an external magnetic field, perpendicular to the broad surface of the slab. As the demagnetizing coefficient of the sheet is nearly unity, the magnetic field penetrates at a very low field value in the form of flux bundles, containing many quanta (type I), or single quantized vortices (type II). A transport current, if present, interacts with the bundles or vortices and exerts a force on them. It is plausible that this force is balanced by a force due to the pinning by structural defects or by a friction force or a combination of both.

In the model to be discussed we assume the absence of pinning centres.

The material inside the flux bundles is fully normal, thus it has the conductivity $\sigma$ and the Hall coefficient $R_H$ of the normal state. As in nearly all cases the diameter of the flux bundle is larger than the mean free path $l$, the use of the transport quantities $\sigma$ and $R_H$ is meaningful in the case of bundles.

In dealing with the electrons inside the core of a vortex in a type-II material we must be more careful. From the calculations of Caroli, De Gennes and Matricon 7) we know that we may consider the core as a cylinder with radius $\xi$, containing normal material. Here $\xi$ is the coherence length of the superconducting state. However, the use for these cylinders of the ordinary transport quanti-

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ties $\sigma$ and $R_H$ is meaningful only when the mean free path is smaller than or of the order of magnitude of the diameter of the vortex core, i.e. $l \leq \xi$. Now the quantity $\xi$ is determined by $\xi_0$, the coherence length of the pure material, and $l$. In the very impure limit, $l \ll \xi_0$, we have $\xi \approx (\xi_0 l)^{1/2}$, and the inequality $l < \xi_0$ obviously is then satisfied. In the pure limit $l \gg \xi_0$, however, we have $\xi \approx \xi_0 \ll l$. For the very pure type-II superconductor the treatment we will present, based on the use of the normal transport equations for the core, becomes somewhat doubtful. Hence we may say that as far as we use the transport equations, our treatment is suitable for the case of flux bundles in type-I materials and also for vortices in very impure type-II superconductors, whereas one should be careful in applying our results to the case of a very pure type-II material.

We assume the external transport-current density $j$ to be uniform throughout the sample. In general the flow pattern belonging to the external current will be distorted by the vortices, giving rise to a backflow inside the core and near its boundary. However, even taking into account the backflow one is left with an extra unknown parameter, namely the angle between the current flow inside the core and far away from the vortex. Recently Nozières and De Gennes \textsuperscript{8}, and Vinen \textsuperscript{9} independently have made models with backflow. They assume the current flow inside to be parallel to the flow far away from the core. Our assumption of the uniform transport current avoids the problem of the unknown parameter and from the final results we get it seems to be a plausible assumption.

Our model means that the current densities inside and outside the cores are equal and that the current in the core is a completely normal one. This picture involves the electrons in a continuous transition from the normal to the superconducting state. The consequences of these transitions will be discussed in this paper.

The system of moving flux lines or normal areas leads to voltages across the specimen which have to be interpreted as resistivity and Hall effect.

2. Calculation of the electric field

2.1. Geometry

Our frame of reference is that of the lattice. We consider a superconducting slab in the $X-Y$ plane. The external magnetic field $H_0$ points in the $z$-direction, while the external current density $j$ is in the $y$-direction, see fig. 1. The unit vectors $\mathbf{1}_x$, $\mathbf{1}_y$ and $\mathbf{1}_z$ are positive-oriented. The vortices or normal areas there will move with a velocity $\mathbf{v}_L$. This velocity $\mathbf{v}_L$ of course has no $z$-component and makes a (negative) angle $\alpha$ with the positive $x$-axis. The vector $\mathbf{v}_L$ is assumed to be constant in space and time.

2.2. The outside region

We now restrict ourselves to type-II materials, so the case of vortices will be...
considered. Let us first consider the outside region. Here the superconducting electrons are subject to a velocity field $v_s = v_s(r, t)$, a magnetic field $H = H(r, t)$, an electric field $E = E(r, t)$ and a gradient in the chemical potential $\mu_{s0}$ of the superfluid component. The concentration of superconducting electrons $n_s$ is also a function of space and time, and of the temperature. However we will neglect the spatial variation in the outside region. This means the Ginzburg-Landau parameter $\kappa$ must be rather large compared to unity.

The hydrodynamic equation of motion for the superfluid reads as (per unit volume)

\[
\frac{d v_s}{dt} = n_s e E + n_s e \frac{v_s \wedge H}{c} + \frac{n_s}{N} \nabla p_0 ,
\]

in which $N = n_s + n_n$, the total number of charge carriers per unit volume. The last term $(n_s/N)\nabla p_0$ equals $-n_s \nabla \mu_{s0}$ and is already present in the stationary case as discussed in ref. 10.

It may be useful to remark that in general one can write for the gradient of $\mu_{s0}$:

\[
\nabla \mu_{s0} = \nabla \mu_{s0}^{(1)} + \nabla \mu_{s0}^{(2)} ,
\]

where $\nabla \mu_{s0}^{(1)}$ is the volume force arising from the spatial variation of the order parameter $n_s/N$ and where $\nabla \mu_{s0}^{(2)}$ comes from the motion of the normal component with respect to the superfluid. In our model we take $n_s$ constant in the outside region and zero inside, thus $\nabla \mu_{s0}^{(1)}$ is zero everywhere except at the core boundary. The force $\nabla \mu_{s0}^{(2)}$ has the character of a relative kinetic pressure gradient and behaves therefore as $\nabla (v_n - v_s)^2$, see for instance Landau and
Lifshitz 11). Physically this force represents collisions of the normal with the superconducting electrons.

Substituting \[ \frac{\partial v_s}{\partial t} = \frac{\partial v_s}{\partial t} + \frac{1}{2} \nabla v_s^2 - v_s \land \text{curl} \, v_s \] in eq. (1) we get

\[ n_s \frac{\partial v_s}{\partial t} = -n_s \nabla v_s^2 + n_s v_s \land \left[ \text{curl} \, m v_s + \frac{e}{c} H \right] + n_s e E + n_s \frac{n_s}{N} \nabla p_0 . \] (2)

The bracket term drops out due to London's equation:

\[ \text{curl} \, m v_s = - \frac{e}{c} H . \]

The total velocity field \( v_s \) can be split up into two parts. One part which we call \( v_{so} \) is related to the external uniform current density \( j \). This leads to a uniform velocity \( v_{so} \). The remaining part \( v_{s1} \) is the circulation field proper of the vortex, and thus

\[ v_s(r, t) = v_{so} + v_{s1}(r, t) . \] (3)

Substituting eq. (3) into eq. (2), the last equation reduces to

\[ m \frac{\partial v_{s1}}{\partial t} = -\frac{m}{2} \nabla (v_{s1}^2 + 2 v_{s1} \cdot v_{so}) + e E + \frac{1}{N} \nabla p_0 . \] (4)

It is also convenient to split up the total electric field into

\[ E = E_0 + E_1 , \]

where \( E_1 \) is defined by

\[ e E_1 = \frac{m}{2} \nabla v_{s1}^2 + m \nabla (v_{s1} \cdot v_{so}) - (1/N) \nabla p_0 . \] (5)

To this electric field \( E_1 \) belongs a potential function \( \psi \) which in any point of the outside region is given by

\[ \psi = \frac{-m}{2e} (v_{s1}^2 + 2 v_{s1} \cdot v_{so}) - \frac{\mu_{so}}{e} , \] (6)

where we now have used \( -\mu_{so} \) for \( p_0/N \), neglecting an irrelevant additive constant. Let us denote the value of \( \psi \) on the boundary of the core by \( \psi_c \), corresponding to \( v_{s1} \) (\( v_{s1} \) taken at the boundary). This potential, however, is counterbalanced by a contact potential between the inner and outer regions, as can be seen as follows.

The electrochemical potential outside the core is given by

\[ \mu_s = \mu_{so} + \frac{1}{2} m v_s^2 + e \psi . \] (7)
Here $\mu_{s0}$ is the chemical potential of the superconducting particles. The other two terms are additions due to current and electric field, respectively. For the normal electrons inside the core we have

$$\mu_n = \mu_{n0} + \frac{1}{2}mv_n^2 + e\psi_n,$$

(8)

where $\psi_n$ is the electric potential inside the core.

At $T = 0$ the uniform current leads to $v_n = v_{s0}$. Substituting this in eq. (8) and taking eqs (7) and (8) at the core boundary we get

$$\mu_{sc} = \mu_{s0} + \frac{1}{2}mv_{so}^2 + e\psi_c,$$

$$\mu_{nc} = \mu_{n0} + \frac{1}{2}mv_{no}^2 + e\psi_{nc}.$$

(9)

In the stationary situation one knows that the normal electrons inside the core and the superconducting electrons in the outside regions must be in equilibrium at the boundary. This involves $\mu_{sc} = \mu_{nc}$ in that case. Our problem deals with a non-stationary situation. As is usually done in irreversible thermodynamics, however, we assume that in first-order approximation the condition of local equilibrium still holds. From $\mu_{sc} = \mu_{nc}$ follows:

$$\mu_{n0} - \mu_{s0} + \frac{1}{2}m(v_{so}^2 - v_{se}^2) + e(\psi_{nc} - \psi_{sc}) = 0.$$

(10)

For $v_{se}^2$ we have

$$v_{se}^2 = v_{so}^2 + v_{slc}^2 + 2(v_{slc} \cdot v_{so}).$$

From these we obtain for the contact potential $\Delta \psi$:

$$\Delta \psi \equiv \psi_{nc} - \psi_{sc} = \frac{\mu_{so} - \mu_{n0}}{e} + \frac{m(v_{slc}^2 + 2v_{slc} \cdot v_{so})}{2e}.$$

(11)

The first term of the right-hand side of eq. (11) is the contact potential needed to compensate the difference in potential energy between the superconducting and normal electrons. The second term arises from the difference in the kinetic energies between the outside and inside regions. We needed these considerations in order to make clear that the electric field $E_1$ does not contribute to the measured field. We now proceed with the electric field $E_0$. Using

$$\frac{\partial v_{sl}}{\partial t} = -(v_L \cdot \nabla)v_{sl},$$

we derive from eq. (4):

$$-m(v_L \cdot \nabla)v_{sl} = eE_0,$$

(12)

in which $v_L$ is the uniform line velocity. It can be seen in eq. (12) that $E_0$ is divergence-free, because the electronic fluid is incompressible, thus $\text{div } v_{sl} = 0$.

On the other hand, from Maxwell's equation we get

$$\text{curl } E = \text{curl } E_0 - \frac{1}{c} \frac{\partial H}{\partial t} = \frac{1}{c} (v_L \cdot \nabla)H = \frac{1}{c} \text{curl } (v_L \wedge H).$$
It can be seen from eq. (5) that \( \text{curl } \mathbf{E}_1 = 0 \), because \( \mathbf{E}_1 \) is a gradient. Thus \( \mathbf{E}_0 \) is given by

\[
\mathbf{E}_0 = -\frac{1}{c} \mathbf{v}_L \wedge \mathbf{H} - \nabla \varphi,
\]

where \( \varphi \) is a function determined by \( \text{div } \mathbf{E}_0 = 0 \). From this condition, again using London's equation, we find

\[
A \varphi = \left( -\frac{1}{c} \right) \nabla \cdot (\mathbf{v}_L \wedge \mathbf{H}) = \frac{m}{e} (\mathbf{v}_L \cdot \Delta \mathbf{s}) = \frac{m}{e} (\mathbf{v}_L \cdot \Delta \mathbf{s}_1) = \frac{m}{e} \Delta (\mathbf{v}_L \cdot \mathbf{s}_1).
\]

Hence we obtain a solution for \( \varphi \):

\[
\varphi = \frac{m}{e} (\mathbf{v}_L \cdot \mathbf{s}_1).
\]

To \( \varphi \) we may add an arbitrary function \( \varphi_1 \) which satisfies \( A \varphi_1 = 0 \). As \( \varphi_1 \) must be finite everywhere, it is a constant and hence contributes nothing to the electric field.

As \( \mathbf{s}_1 \) is a function of space and time, \( \varphi \) is also space- and time-dependent. In order to give an explicit expression for \( \varphi \) as a function of \( x \) and \( y \) we consider \( \varphi \) at a certain instant of time, namely at the instant when the centre of a certain vortex just passes the origin of the coordinate system. Writing

\[
\varphi = \frac{m}{e} (\mathbf{v}_L \cdot \mathbf{s}_1) = \frac{m}{e} \mathbf{v}_L \mathbf{s}_1 \cos \beta,
\]

one can use the quantization condition to get an expression for \( \mathbf{s}_1 \) in the following way:

\[
\oint \frac{2m \mathbf{v}_0 \cdot d\mathbf{l}}{r} + \oint \frac{2e}{c} \mathbf{A} \cdot d\mathbf{l} = \oint \frac{2m \mathbf{s}_1 \cdot d\mathbf{l}}{r} + \oint \frac{2e}{c} \mathbf{A} \cdot d\mathbf{l} = h.
\]

Here \( \Gamma \) is a contour of integration with radius \( r \) going just round the vortex under consideration. If we write

\[
\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{H} \cdot d\mathbf{S} = -\Phi(H_0, r),
\]

we obtain for \( \mathbf{s}_1 \):

\[
\mathbf{s}_1 = \frac{1}{4\pi mr} \left[ h + \frac{2e}{c} \Phi(H_0, r) \right].
\]

We have introduced the quantity \( \Phi \), the magnetic flux contained in a loop \( \Gamma \). It is obviously a function of \( r \) and the external magnetic field \( H_0 \). The minus sign in the definition comes from the fact that \( d\mathbf{l} \) has been taken in the direction of \( \mathbf{s}_1 \) whereas the vector \( \mathbf{A} \) circulates in opposite direction. It may be useful to point out that \( \mathbf{s}_1 \) from eq. (16) is zero if we take \( r = d/2 \), that is half the distance between neighbouring vortices for all field values between \( H_{c_1} \) and \( H_{c_2} \).
With fig. 2 it is easy to calculate $\cos \beta$, occurring in eq. (15), and inserting this, together with eq. (16) we get for $\varphi$:

$$
\varphi = \frac{v_L}{4\pi e r^2} \left[ h + \frac{2e}{c} \Phi (H_0, r) \right] \left[ y \cos \alpha - x \sin \alpha \right].
$$

(17)

For the electric field $E_0$ (see eq. (13)) we need the gradient of $\varphi$. In polar coordinates $(r, \delta)$ we find

$$
\nabla \varphi = \frac{v_L}{4\pi e} \mathbf{1}_r \left[ \frac{2e}{c} \frac{1}{r} \frac{d}{dr} \left( h + \frac{2e}{c} \Phi \right) \right] \sin (\delta - \alpha) +
$$

$$
+ \frac{v_L}{4\pi e} \mathbf{1}_s \frac{1}{r^2} \left( h + \frac{2e}{c} \Phi \right) \cos (\delta - \alpha).
$$

(18)

It is convenient to describe $\varphi$ in terms of the components along $v_L$ and perpendicular to $v_L$ as indicated in fig. 2. In this way is derived:

$$
\nabla \varphi = \frac{v_L}{4\pi e} \mathbf{1}_{v_L} \left[ \frac{2e}{c} \frac{1}{r} \frac{d}{dr} \sin (\delta - \alpha) \cos (\delta - \alpha) \right] +
$$

$$
- \frac{v_L}{4\pi e} \mathbf{1}_{v_L} \left[ \frac{1}{r^2} \left( h + \frac{2e}{c} \Phi \right) \sin 2(\delta - \alpha) \right] +
$$

$$
+ \frac{v_L}{4\pi e} \mathbf{1}' \left[ \frac{1}{r^2} \left( h + \frac{2e}{c} \Phi \right) \cos 2(\delta - \alpha) \right] +
$$

$$
+ \frac{v_L}{4\pi e} \mathbf{1}' \left[ \frac{2e}{c} \frac{1}{r} \frac{d}{dr} \sin^2 (\delta - \alpha) \right].
$$

(19)

The electric field $E_0$ in the outside region is thus completely determined.

Fig. 2. Schematic diagram for the transformation of formula (18) to formula (19).
2.3. Inside region

Turning to the problem of the determination of the electric field in the core, it should be noted that the electric field $E_1$, defined in eq. (5), has no continuation inside the core of a vortex, because this field is counterbalanced by the contact potential as stated above. From eq. (17) it is clear that the potential $\varphi$ at the boundary of the core ($r = \xi$) takes the value

$$\varphi = \frac{v_L}{4\pi e} \frac{1}{\xi^2} \left[ h + \frac{2e}{c} \Phi (H_0, \xi) \right] \left( y \cos a - x \sin a \right). \quad (20)$$

The potential $\varphi$ must be continuous across the boundary. Besides this, it was stated in the introduction that the cores are considered normal, hence a linear relationship between current density and electric field must apply, and the current density, too, was assumed to be uniform. Thus the electric field inside the cores must be also uniform and eq. (20) represents the potential in the core itself. Now for the inside electric field $E_t$ we have

$$E_t = (-1/c) v_L \wedge H_k - \frac{v_L}{4\pi e} \frac{1}{\xi^2} \left[ h + \frac{2e}{c} \Phi (H_0, \xi) \right] (-l_x \sin a + l_y \cos a).$$

By $H_k$ we mean the core field. This can be written in a more transparent form. First:

$$-\frac{1}{c} v_L \wedge H_k = l' \frac{v_L H_k}{c};$$

second:

$$-\frac{h}{4\pi e \xi^2} = \frac{1}{2c} \frac{\phi_0}{\pi \xi^2} = \frac{H_{c_2}}{2c}, \quad \left( \phi_0 = \frac{hc}{2 |e|}, \text{ the flux unit} \right);$$

third:

$$\Phi(H_0, \xi) = H_k \pi \xi^2, \quad \text{so} \quad \frac{1}{2\pi \xi^2} \frac{1}{c} \Phi = \frac{H_k}{2c};$$

and finally

$$-l_x \sin a + l_y \cos a = -l'.$$

Thus

$$E_t = l' \left( \frac{v_L}{2c} \right) [H_k(H_0) + H_{c_2}]. \quad (21)$$

As we see, $E_t$ is perpendicular to the velocity $v_L$.

2.4. The averaged electric field

From the local fields as given by eqs (13), (19) and (21) we have to calculate
the averaged electric field, because it is this field which is measured in an experimental set-up. The averaged field is defined by

\[ E_{av} \equiv \frac{1}{d^2} \left[ \int_{S_1=\pi \xi^2} E_t \, dx \, dy + \int_{S_2=d^2-\pi \xi^2} E_0 \, dx \, dy \right], \tag{22} \]

where \( d \) is the distance between vortices. As has already been pointed out, the field \( E_1 \) does not contribute to the averaged field. Returning now to the calculation of \( E_{av} \), we first consider the contribution from \( E_0 = 1'v_L H/c - \nabla \varphi \). It is clear from eq. (19) that only the last term in this equation makes a contribution, namely

\[ -1' \frac{v_L}{2\pi c} \frac{1}{d^2} \int_{S_1} \frac{1}{r} \frac{d\Phi}{dr} \sin^2(\delta - \alpha) \, dx \, dy. \tag{23} \]

We approximate this by integrating over a circle of radius \( d/2 \) instead of over a square of dimension \( d \). This is certainly a good approximation if \( H_0 \) is not too close to \( H_{c2} \). We then obtain for the expression (23):

\[ -1' \frac{v_L}{2c} \frac{1}{d^2} \left\{ \Phi(H_0, d/2) - \Phi(H_0, \xi) \right\} = -1' \frac{v_L}{2c} \frac{1}{d^2} \left[ \phi_0 - \Phi(H_0, \xi) \right]. \tag{24} \]

Combining the remaining contribution from \( E_0 \) and the term \( v_L H_k/c \) in \( E_t \), we derive

\[ 1' \frac{v_L}{c} \frac{1}{d^2} \left\{ \int_{S_1} H_k \, dx \, dy + \int_{S_2} H \, dx \, dy \right\} = 1' \frac{v_L B(H_0)}{c}. \tag{25} \]

We still have from \( E_t \):

\[ 1' \frac{v_L}{2c} \frac{1}{d^2} \int_{S_1} (H_{c2} - H_k) \, dx \, dy = \]

\[ = 1' \frac{v_L}{2c} \frac{1}{d^2} \left[ H_{c2} \pi \xi^2 - \Phi(H_0, \xi) \right] = \]

\[ = 1' \frac{v_L}{2c} \frac{1}{d^2} \left[ \phi_0 - \Phi(H_0, \xi) \right]. \tag{26} \]

We see that the contributions given in eqs (24) and (26) cancel each other exactly, and that the averaged field is given by

\[ E_{av} = \frac{1' v_L B(H_0)}{c}. \tag{27} \]

If we compare this final result with the general expression for the local field \( E = (-1/c)v_L \wedge H - \nabla \varphi \) we see that the gradient term does not contribute to
the averaged field. And this in turn comes from the rather surprising effect that the integration of gradient $\varphi$ over the outside region exactly cancels the integration over the inside region.

We end this section with some remarks on the origin and measurability of the electric field $E_{av}$. The right-hand side of eq. (27) suggests we should consider the averaged field as arising from the transport of flux from one side of the sample to the other into the circuit containing the volt meter. However, the measuring circuit in the experimental set-up always contains a constant flux. Therefore it is wrong to interpret the measured voltage as being an inductive E.M.F. in a closed loop.

What is then the origin of the measured voltage? We believe the following to be the answer. When the vortices move through the sample as indicated in fig. 3, real space charges are induced on the boundary of the cores. A set of moving cores is thus equivalent to a set of moving electric dipoles and our calculated averaged field $E_{av}$ is indeed the averaged field of one dipole. Of course, this field can be measured. The dipole character of the moving cores arises from the fact that, although the potential function $\varphi$ is continuous across the boundary of the core, the gradient $\varphi$ makes a jump. And thus there is a jump between $E_i$ and the $J'$-component of $E_0$, as one can see by comparing the gradient $\varphi$ in eq. (20) and formula (19).

Thus there must be a space charge on the boundary of the core which fills up the jump in the electric field. Therefore the origin of the measured electric field is the same as that of the Hall field in an ordinary Hall specimen.

Our definition of $E_{av}$ implies that all dipoles are the same. This, of course, is true for an infinite plane slab. In a finite slab, however, the boundary conditions come in due to the mirror effect. We do not take into consideration these effects, because only in a thin layer of thickness $\lambda$ (the penetration length) near the edges of the sample do the electric dipoles differ from those in a central part of the specimen.

Fig. 3. Experimental arrangement of a superconducting specimen and a volt meter. The motion of the vortices is as sketched ($v_L$), giving rise to space charges ($+$ and $-$).
3. Equilibrium of forces

The velocity $v_L$ of the vortex appears in the expression of the inside and averaged electric fields. The next problem is thus to determine $v_L$. This can be done by using a relation which describes the equilibrium of forces acting on a moving vortex. We now turn to this problem. In the following we always consider a unit length of the vortex and ideal specimens—no pinning—as had been said in the introduction. Recently a lot of discussion concerning the forces on a moving vortex has emerged and now at least three models have been put forward.

(a) In the first model the driving force is the ordinary Ampère force $n_s e v_{s0} \wedge (\phi_0/c)\hat{1}_z$. This force is then balanced by the Magnus force $K_M = (n_s e \phi_0/c)v_L \wedge \hat{1}_z$ and a damping force $f v_L$, where $f$ is a scalar damping coefficient. Thus the force equation here reads as

$$(n_s e \phi_0/c) v_{s0} \wedge \hat{1}_z = (n_s e \phi_0/c) v_L \wedge \hat{1}_z + f v_L .$$

From this relation one sees that $v_L$ is perpendicular to $(v_{s0} - v_L)$, see fig. 4. The angle $\gamma$ between $v_{s0}$ and $v_L$ is a function of $f$. For $f = 0$ it follows that $\gamma = 0$ and $v_L = v_{s0}$, so then the vortex moves with the drift flow, leading to a voltage in the transverse direction only. For $f \gg n_s|e|\phi_0/c$, $v_L$ is nearly perpendicular to $v_{s0}$ and the occurring voltage is almost longitudinal. For intermediate values of $f$ longitudinal as well as transverse voltages appear, hence a Hall angle exists.

The above model was used by many authors $^{13,14)}$.

(b) The second model states that the driving force, again being the Ampère force is balanced by the damping force $f v_L$ only. In this model which has been presented by the authors of refs 1, 2, 4, 15 and 16, the idea of a Magnus force is rejected for reasons which were partly of experimental nature $^{1,2,4)$, partly theoretical $^{15)}$. As already has been said above in this second model no Hall effect can exist, because $v_L$ is perpendicular to $v_{s0}$, see fig. 4.

(c) Third: recently independent calculations of Vinen $^2$) and Nozières and De Gennes $^8$) have put forward a force on the core of the vortex which force was omitted both in the first and second models. The above-mentioned authors

![Fig. 4. Schematic diagram for the drift velocity $v_{s0}$ and the line velocity $v_L$ occurring in the models sub (a) (full line) and sub (b) (dashed line).](image-url)
pointed out that an electron makes a jump in mechanical momentum when it moves from the outside region into the core of the vortex. This stems from the fact that an electron in the outside region is subjected to the circulation velocity of the vortex itself whereas inside the core this circulation velocity is zero. This discontinuity in the momentum corresponds to a force exerted by the outside electrons on the core.

At the time when the model mentioned in (b) was published no reliable experimental evidence for a Hall effect was available. In the meantime, however, Niessen and Staas 6) *) have published experimental verification of a genuine Hall effect. This means that the model sub (b) cannot be right in general. On the other hand the model sub (a) predicts \( v_L \leq v_{so} \) (see fig. 4), so the longitudinal electric field \( E_{||} \) should have a maximum value. This maximum in \( E_{||} \), however, has not been found experimentally. In view of this situation we have constructed a new model. It will come out to stand in between the models sub (a) and (b) in the sense that it results in having a Magnus-type force but with a reduced magnitude, as will be shown now.

We present this model first without considering the force due to the transfer of momentum 18) as indicated in the model sub (c) and later on we will include this force.

3.1. Force equation excluding the transfer of momentum

Our reasoning goes along the following lines. It is well known that in the normal state the relation between current density and electric field is governed by a conductivity tensor \( \sigma \) rather than by a scalar quantity, if a magnetic field is present. As was suggested to us by Volger, this means that in our case the damping coefficient \( f \) also will be a tensor. Taking a tensor for \( f \) is in fact an extension of the model presented in the paper mentioned under ref. 13 and as we shall see later on, it yields a reasonable Hall effect.

Thus we put

\[
\mathbf{f} v_L = \mathbf{j} \wedge \frac{\mathbf{e}_0}{c} \mathbf{1}_z. \tag{28}
\]

In metals with cubic symmetry it is possible to split up the tensor \( f \) into a symmetric part \( f^{(s)} \) and an antisymmetric part \( f^{(a)} \), where

\[
f^{(s)} = \begin{pmatrix}
  f_1 & 0 & 0 \\
  0 & f_1 & 0 \\
  0 & 0 & f_1
\end{pmatrix}.
\]

In our geometry \( f^{(a)} \) takes the form:

\[
f^{(a)} = \begin{pmatrix}
  0 & f_2 & 0 \\
  -f_2 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]

*) Note added in proof. The existence of Hall effect has now been verified too by Ree et al. 17).
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We now proceed to determine $f_1$ and $f_2$. As the vector $v_L$ only has components in the $X$-$Y$ plane, eq. (28) reduces to

$$f_1 v_{Lx} + f_2 v_{Ly} = j \phi_0/c$$

and

$$f_1 v_{Ly} - f_2 v_{Lx} = 0.$$  

From this we derive

$$v_{Lx} = \frac{j \phi_0}{c} \frac{f_1}{f_1^2 + f_2^2}$$  \hspace{1cm} (29)  

and

$$v_{Ly} = \frac{j \phi_0}{c} \frac{f_2}{f_1^2 + f_2^2}.$$  \hspace{1cm} (30)

It is to be seen that where $|f_2|$ is negligibly small compared to $f_1$, $v_{Ly}$ is nearly zero, so the vortices move perpendicular to the current. For the case $f_1 \ll |f_2|$ the vortices move nearly parallel to the external current. For the inside region we know the electric field $E_t$ (eq. (21)), and we also have an expression for the line velocity $v_L$ in terms of $f_1$ and $f_2$ (eqs (29) and (30)), so we now are able to connect $f_1$ and $f_2$ with the parameters $\sigma$ and $R_H$ of the normal region. Using $E_{t\|}$ and $E_{t\perp}$ as components of the inside electric field parallel and perpendicular, respectively, to the transport current, we can write

$$E_{t\|} = \frac{1}{2c} \left[ H_k + H_{c2} \right] v_L \cos \alpha =$$

$$= \frac{1}{2c} \left[ H_k + H_{c2} \right] v_{Lx} = \frac{j \phi_0}{2c^2} \left[ H_k + H_{c2} \right] \frac{f_1}{f_1^2 + f_2^2}.$$  \hspace{1cm} (31)

In this region the relation $j = \sigma E_{t\|}$ is valid, leading to

$$\frac{\sigma \phi_0}{2c^2} \left[ H_k + H_{c2} \right] \frac{f_1}{f_1^2 + f_2^2} = 1.$$  \hspace{1cm} (31)

A second equation containing $f_1$ and $f_2$ is derived in an analogous way:

$$E_{t\perp} = -\frac{1}{2c} \left[ H_k + H_{c2} \right] v_L \sin \alpha = -\frac{j \phi_0}{2c^2} \left[ H_k + H_{c2} \right] \frac{f_2}{f_1^2 + f_2^2}$$

and

$$E_{t\perp} = -R_H j H_k \text{ (Hall effect in the core), which together give}$$

$$\frac{\phi_0}{2 R_H c^2} \left[ H_k + H_{c2} \right] \frac{f_2}{f_1^2 + f_2^2} = 1.$$  \hspace{1cm} (32)
From eqs (31) and (32) it is easy to solve \( f_1 \) and \( f_2 \). The result is

\[
f_1(H_0) = \frac{\phi_0}{2c^2} \frac{H_k(H_0) + H_{c2}}{1 + [\sigma R_H H_k(H_0)]^2} \tag{33}
\]

and

\[
f_2(H_0) = \frac{\sigma^2 \phi_0 R_H}{2c^2} \frac{H_k(H_0)[H_k(H_0) + H_{c2}]}{1 + [\sigma R_H H_k(H_0)]^2} . \tag{34}
\]

It is instructive to make a plot of \( f_1 \) and \( f_2 \) as a function of \( \sigma \) for some fixed value of the external magnetic field, see fig. 5. It is clear that we have to distinguish between two different regimes, determined by \( \sigma R_H H_k \ll 1 \) much less than unity and much greater than unity, respectively.

Let us first consider the region \( \omega_c(H_k) \tau \ll 1 \). This condition is somewhat inconvenient because it depends on \( H_k \) and thus on \( H_0 \). We therefore use the condition \( \omega_c(H_{c2}) \tau \ll 1 \) which certainly implies \( \omega_c(H_k) \tau \ll 1 \). This is the regime which will apply to nearly all type-II superconductors, because these superconductors are often alloys of transition metals. In this region \( f_1 \) increases linearly with \( \sigma \) and \( f_2 \) quadratically. But as we always have \( |f_2| = f_1 \omega_c(H_k) \tau \), \( |f_2| \) is a few orders of magnitude smaller than \( f_1 \). This implies (eqs (29) and (30)) \( |v_{Lz}| \ll v_{Lx} \) or stated in other words, that the vortices move nearly perpendicular to the current.

In the limit under discussion \( f_1 \) (eq. (33)) reduces to

\[
f_1(H_0) = \frac{\phi_0}{2c^2} \left[ H_k(H_0) + H_{c2} \right] .
\]

If \( H_0 \) reaches \( H_{c2} \) one finds for \( f_1 \):

\[
f_1 \approx \frac{\phi_0}{c^2} H_{c2}, \quad H_0 \approx H_{c2} . \tag{35}\]
This is precisely the same formula as that given by Strnad et al. 2) on empirical grounds and derived theoretically by Stephen and Bardeen 15). It is surprising, that our expression for \( f_1 \) near \( H_{c2} \) agrees with the empirical results, because it is very doubtful whether our calculation is correct close to the upper critical field.

For values of \( H_0 \) close to \( H_{c1} \) the approximate expression is

\[
f_1 \approx \frac{\sigma \phi_0}{2c^2} H_{c2}, \quad H_0 \approx H_{c1}.
\]

The analogous expressions, \( \omega_c(H_{c2}) \tau \ll 1 \), for \( f_2 \) are

\[
f_2(H_0) = \frac{\sigma^2 \phi_0 R_H}{2 c^2} H_k(H_0) \left[ H_k(H_0) + H_{c2} \right],
\]

\[
f_2 \approx \frac{\sigma^2 \phi_0 R_H}{c^2} H_{c2}^2, \quad H_0 \approx H_{c2}, \quad (38)
\]

\[
f_2 \approx \frac{\sigma^2 \phi_0 R_H}{c^2} H_{c1} H_{c2}, \quad H_0 \approx H_{c1}. \quad (39)
\]

Next we consider the region \( \omega_c(H_{c2}) \tau \gg 1 \). This condition is equivalent to \( l \gg \xi \approx \xi_0 \). As we have pointed out in the introduction, our treatment, based on the use of the normal transport equations inside the core, should be taken with some care. For \( f_1 \) and \( f_2 \) we find

\[
f_1(H_0) = \frac{\phi_0}{2c^2} \frac{1}{\sigma R_H^2} \frac{H_k(H_0) + H_{c2}}{[H_k(H_0)]^2}
\]

and

\[
f_2(H_0) = \frac{\phi_0}{2c^2} \frac{1}{R_H} \frac{H_k(H_0) + H_{c2}}{H_k(H_0)}.
\]

As indicated in fig. 5, the constant value of \(|f_2|\) (as a function of \( \sigma \)) equals two times the maximum value of \( f_1 \), where the maximum is reached in \( \sigma_{\text{max}} = 1/|R_H|H_k \), corresponding to \( \omega_c(H_k) \tau = 1 \).

We see that in the limit now under discussion, \(|f_2|\) is a few orders of magnitude larger than \( f_1 \). At first sight the behaviour of \( f_1 \) as a function of \( \sigma \) beyond the maximum is unexpected, because it is customary to expect that the damping increases as the conductivity \( \sigma \) increases. However, it should be realized that in the range governed by \( \omega_c(H_{c2}) \tau \gg 1 \), the normal electrons describe many cyclotron orbits within the relaxation time \( \tau \). This means the electrons then do not pick up much effective momentum and energy from the electric field and so do not transfer much momentum and energy to the lattice. Although the region \( \omega_c(H_{c2}) \tau \gg 1 \) is not reached by many superconductors of the second kind, we
believe that this region can be important for type-I superconductors, where there are normal moving spots. In the paper under ref. 6 longitudinal and transverse voltages, measured on a pure lead sample, were also reported.

We end the discussion of \( f_1 \) and \( f_2 \) by remarking that for \( f^{(a)} v_L \) one can write \( f^{(a)} v_L = -f_2 \mathbf{1}_z \wedge v_L \). Hence this force on the moving vortex is of the same structure as the Magnus force. Therefore our description using a tensor for \( f \) does not avoid the concept of a Magnus force, strictly speaking. However, the orders of magnitude of the Magnus force in the usual form \( K_M = (n_s e \phi_0 / c) v_L \wedge \mathbf{1}_z \) and our force \( f^{(a)} v_L = -f_2 \mathbf{1}_z \wedge v_L \) are completely different in the case of \( \omega_c(H_{c2}) \tau \ll 1 \). To establish this, it is convenient to consider both forces at \( T = 0 \)°K as this simplifies the discussion. Taking for \( f_2 \) the expression (37) and \( n_s = N \) we find for the ratio of the two forces

\[
\frac{f_2 \mathbf{1}_z \wedge v_L}{(n_s e \phi_0 / c) v_L \wedge \mathbf{1}_z} = \frac{1}{2} \omega_c(H_k) \tau \left[ \omega_c(H_k) \tau + \omega_c(H_{c2}) \tau \right],
\]

which reduces to

\[
\{\omega_c(H_{c2}) \tau\}^2 \quad \text{for } H_0 = H_{c2}
\]

and which is always much smaller than unity.

Where \( \omega_c(H_{c2}) \tau \gg 1 \) we find from eq. (40):

\[
\frac{|f_2 v_L \wedge \mathbf{1}_z|}{|(n_s e \phi_0 / c) v_L \wedge \mathbf{1}_z|} = \frac{H_k + H_{c2}}{2 H_k}.
\]

As we already stated, the condition \( \omega_c(H_{c2}) \tau \gg 1 \) in general is not fulfilled for type-II superconductors. However, it may apply to pure type-I materials, where the condition then reads \( \omega_c(H_c) \tau \gg 1 \). Here \( H_c \) is the thermodynamic critical field. In that case the ratio (42) is exactly unity and for type-II materials which satisfy \( \omega_c(H_{c2}) \tau \gg 1 \) the ratio will be very close to unity in practical cases. However, as the concept of the Magnus force certainly is a sound concept in pure superconductors the obtained ratio (42) indicates that our treatment is somewhat inadequate for the case of very pure type-II materials.

In the papers under ref. 5 the authors used an expression for the Magnus force of the form

\[
(\theta n_s e \phi_0 / c) v_L \wedge \mathbf{1}_z,
\]

where \( \theta \) was an undetermined parameter. We can now give an explicit expression for this parameter \( \theta \), namely

\[
\theta = \frac{\frac{1}{2} \omega_c(H_k) \tau \left[ \omega_c(H_k) \tau + \omega_c(H_{c2}) \tau \right]}{1 + \left[ \omega_c(H_k) \tau \right]^2}.
\]

Hence we may summarize by saying that in our model the Magnus force is to be found both in pure and impure materials, but that in impure materials it is strongly reduced in magnitude.
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3.1.1. Flow resistivity

From expression (27) we obtain directly
\[ E_{av\parallel} = \frac{B}{c} v_L a = \frac{j \phi_0}{c^2} B \frac{f_1}{f_1^2 + f_2^2} = \frac{2B \rho j}{H_k + H_{c2}}. \] (43)

From this it follows that the flow resistivity \( \rho_f \) reads as
\[ \rho_f(H_0) = \frac{2 \rho B(H_0)}{H_k(H_0) + H_{c2}}. \] (44)

In the Meissner region \( \rho_f \) is zero and beyond this region \( \rho_f \) increases more or less linearly with \( H_0 \). The detailed dependence of \( \rho_f \) on \( H_0 \) is unknown due to the lack of an explicit expression for \( H_k \) as a function of the external magnetic field. This too makes a detailed comparison with the experiments difficult. The infinite slope at the value \( H_0 = H_{c1} \) predicted by eq. (44) is often not found in measurements, due perhaps to pinning effects.

3.1.2. Hall effect and Hall angle

For the averaged transverse electric field we obtain by analogy with sec. 3.1.1,
\[ E_{av\perp}(H_0) = 2j |R_H| \frac{B(H_0) H_k(H_0)}{H_k(H_0) + H_{c2}}. \] (45)

In the Meissner region the Hall voltage is zero and for \( H_0 \) approaching \( H_{c2} \), \( E_{av\perp} \) takes the value of the Hall voltage in the normal state, whereas in between it increases with the external magnetic field, though its value is always less than the one in the normal state at the same magnetic-field value.

By combination of eqs (43) and (45) we obtain for the Hall angle
\[ |\tan \alpha| = \frac{E_{av\perp}}{E_{av\parallel}} = \sigma |R_H| H_k(H_0). \] (46)

Although we do not know the expression for \( H_k \) as a function of \( H_0 \) we know from the paper of Abrikosov \(^{10}\) that \( H_k \) is always larger than \( H_0 \) up to \( H_0 = H_{c2} \). This means that the Hall angle as predicted by eq. (46) is always larger than the extrapolated value in the normal state which is in agreement with the experimental results of ref. 6.

3.1.3. Energy balance

As the dissipation of energy takes place only in the core the following equation should apply:
\[ d^2 E_{av} \cdot j = \pi \xi^2 E_t \cdot j. \] (47)
Inserting the formulae for $E_a$ and $E_t$ eq. (47) leads to

$$1 = \frac{1}{2} + \frac{1}{2} \frac{H_k}{H_c}. $$

Hence the energy balance fits only at $H_0 = H_c$ and the discrepancy increases with decreasing $H_0$. We come back to this point in the next section.

3.2. Force equation including the transfer of momentum

As is clear from the treatment of sec. 3.1 the force equation (28) fails to describe fully the situation in two respects; it does not fit the energy balance for all values of $H_0 \neq H_c$, and it does not give the correct magnitude of the Magnus force for the pure type-II materials, i.e. when $\omega_c(H_c) \tau \gg 1$, again for all values $H_0 \neq H_c$.

We now proceed to the calculation of the respective quantities mentioned in sec. 3.1, starting from a force equation analogous to (28), but including an extra term due to the transfer of momentum.

As has been stated by the authors of refs 8 and 9 there is a discontinuity in the velocity of an electron moving from the outside region into the core of a vortex. In our model where the drift velocities outside and inside are equal and uniform the extra velocity of the electrons in the outside region is just the circulation velocity $v_{s1}(r)$ of the vortex itself. Hence the transfer of mechanical momentum from the outside region into the core per unit time and per unit length of the vortex is given by (at $T = 0, n_s = N$)

$$\frac{dp}{dt} = F_{tr} = \int \{ (v_L - v_{s0}) \cdot dS \} N m v_{s1c}. \tag{48}$$

Here $dS$ is a vector perpendicular to a surface element $dS$ of the surface of the core, and $dS$ points into the outside region. The quantity $v_{s1c}$ is the circulation velocity at the core boundary. The calculation of the integral yields

$$F_{tr} = \frac{N e \phi_0}{2c} (v_{s0} - v_L) \cdot 1_z \left(1 - \frac{H_k}{H_c}\right), \tag{49}$$

where the last factor $(1 - H_k/H_c)$ comes from

$$2 m \int v_{s1c} \cdot dl = 2 m v_{s1c} \cdot 2 \pi \xi = h + \frac{2e}{c} H_k \pi \xi^2.$$

This force is zero at $H_0 = H_c$ and increases with decreasing external field. This behaviour is physically clear because the circulation velocity at any point $r$ decreases as the vortices come closer to each other, so the discontinuity at the core boundary must decrease with increasing $H_0$.

With eq. (49) the revised force equation now is

$$f' v_L = \frac{j \cdot \phi_0}{c} 1_z + \frac{N e \phi_0}{2c} (v_{s0} - v_L) \cdot 1_z \left(1 - \frac{H_k}{H_c}\right). \tag{50}$$
To avoid confusion with the notation in eq. (28) we have now given the damping tensor a prime.

We now can repeat all steps of the calculation of sec. 3.1. One should notice, however, that the inclusion of $F_{tt}$ in the force equation does not alter the magnitudes of $v_{La}$ and $v_{Ly}$, which enter into the components of the fields $E_t$ (eq. (21)) and $E_{av}$ (eq. (27)). This is so because the components $E_{av\parallel}$ and $E_{av\perp}$ of the electric field inside the normal core are just determined by the transport current $j$ and the external field $H_0$. Thus unaffected by the inclusion of $F_{tt}$ are the quantities $E_{av\parallel}$ and $E_{av\perp}$ and hence the flow resistivity $\rho_f$ and the Hall angle $\alpha$. The altered quantities are the coefficients $f_1$ and $f_2$, denoted therefore by $f_1'$ and $f_2'$ and hence the ratio of our Magnus-type force to the conventional one $K_M = (Ne/e)v_L \wedge \phi_0 \mathbf{l_z}$. The inclusion of $F_{tt}$ affects also the energy equation. We consider the altered quantities now.

The calculations which have to be done are now rather more complicated, but proceed along the same lines as in sec. 3.1. We therefore give only the final results. We obtain for $f_1'$ and $f_2'$:

$$f_1' = \frac{(\sigma/2c^2)[H_k + H_{c2}][3/2 - H_k/2H_{c2}]}{1 + [\sigma R_H H_k]^2}$$

and

$$f_2' = \frac{(\sigma^2/2c^2)H_k[H_k + H_{c2}][3/2 - H_k/2H_{c2}]}{1 + [\sigma R_H H_k]^2} - \frac{Ne\phi_0}{2c}[1 - H_k/H_{c2}],$$

where $f_1'$ and $f_2'$ depend on the external field $H_0$ via the core field $H_k$.

We have plotted these quantities as a function of the conductivity $\sigma$ again for some value of $H_0$, see fig. 6. Comparison of figs 5 and 6 shows that the structure of the curves of fig. 6 is the same as those in fig. 5. There is again a maxi-
mum in $f'_2$ reached at $\omega_c(H_k) \tau = 1$, whereas the curve for $|f'_2|$ versus $\sigma$ is now shifted over an amount $\epsilon$. Here $\epsilon$ stands for $\epsilon(H_0) = (N/|e| \phi_0/2c) [1 - H_k(H_0)/H_c^2]$. The force equation (50) contains two terms which have the structure of a Magnus force, namely $-f'_2 I_z \wedge v_L$ and $(\phi_0 Ne/2c)(v_L \wedge I_z) (1 - H_k/H_c)$. Inserting the expression (52) for $f'_2$ the total Magnus-like force $F'_M$ thus reads now:

$$F'_M = \frac{\sigma^2 R_H \phi_0 H_k (H_k + H_c^2) (3/2 - H_k/2H_c^2)}{2 c^2 [1 + (\sigma R_H H_k)^2]} v_L \wedge I_z.$$

(53)

We again consider the ratio of this force to $K_M = (Ne \phi_0/c) v_L \wedge I_z$ in the two limits $\omega_c(H_c^2) \tau$ much less and much greater than unity. In the first-mentioned limit we get:

$$\frac{|F'_M|}{|K_M|} = \frac{1}{2} \frac{\omega_c(H_k) \tau + 3}{2 \omega_c(H_c^2) \tau} \left[ \frac{\omega_c(H_k) \tau}{2 \omega_c(H_c^2) \tau} \right].$$

(54)

This ratio is much less than unity and it reaches $[\omega_c(H_c^2) \tau]^2$ when $H_0$ goes to $H_c^2$. In the limit of very pure superconductors we get:

$$\frac{|F'_M|}{|K_M|} = \frac{\omega_c(H_k) \tau + \omega_c(H_c^2) \tau}{2 \omega_c(H_c^2) \tau} \left[ \frac{3}{2} \frac{\omega_c(H_k) \tau}{2 \omega_c(H_c^2) \tau} \right].$$

(55)

If we compare eq. (55) with eq. (42) we see that the inclusion of the momentum-transfer force does not lead to the expected magnitude of the Magnus force in very pure type-II superconductors. Only at the upper critical field $H_0 = H_c^2$ does the ratio become unity whereas it should be unity in the whole mixed-state interval. In sec. 3.1 we have already given some criticism about our derivation in the limit of pure type-II superconductors. The criticism formulated there is not stringent for very pure type-I superconductors and indeed we obtain in this case that the ratio (55) is unity because the core field $H_k$ in the intermediate state is equal to the thermodynamic field $H_c$.

3.2.1. Energy balance

Instead of eq. (47) the energy balance now looks like:

$$d^2 E_{av} \cdot j = \pi \varepsilon^2 E_i \cdot j + F_{tr} \cdot v_L.$$

(56)

Here the last term represents the work per unit time done by the force due to the momentum transfer. Substituting for $E_{av}$, $E_i$ and $F_{tr}$ eqs (27), (21) and (49), respectively, we find that the energy balance now is exactly fulfilled. At this stage we would like to mention the damping mechanism introduced recently by Tinkham 20). He pointed out that due to a phase lack in the pairing and depairing process of an electron being alternately in the normal and superconducting states energy is dissipated. One would think that taking into account this dissipation is the same as including in our model the work done by the momentum...
transfer force, although at present we do not know how to transform one mechanism into the other. But both have at least in common to be important at low field values and to go to zero if $H_0$ approaches $H_{c2}$.

4. Comparison with the experiment

From eq. (46) we obtain

$$\frac{|\tan \alpha|_M}{|\tan \alpha|_N} = \frac{H_k(H_0)}{H_0},$$

(57)

where $M$ and $N$ refer to the mixed and the normal state, respectively. From the experiments done on annealed Nb$_{50}$Ta$_{50}$\cite{21} we know experimentally the ratio $|\tan \alpha|_M/|\tan \alpha|_N$, see fig. 7. Hence from this curve we calculated $H_k$ as a function of $H_0$.

![Fig. 7. The ratio of the Hall angle in the mixed state $(E_{||}/E_{||})_M$ to the extrapolated Hall angle $(E_{||}/E_{||})_N$ in the normal state as a function of the external field $H_0$.](image)

Fig. 7. The ratio of the Hall angle in the mixed state $(E_{||}/E_{||})_M$ to the extrapolated Hall angle $(E_{||}/E_{||})_N$ in the normal state as a function of the external field $H_0$.

The values of $H_k$ obtained in this way are larger than $H_{c2}$ in the whole mixed-state interval, which of course is physically far from clear. By inserting, however, the values of $H_k(H_0)$ into the formula for the longitudinal electric field we can make a useful check on the consistency of our model.

From eq. (43) we derive

$$\frac{(E_{av||})_M}{(E_{av||})_N} = \frac{R}{R_N} = \frac{2 B(H_0)}{H_k(H_0) + H_{c2}}.$$  

(58)

As the demagnetizing coefficient of a sheet is nearly unity we may substitute for $B(H_0) = H_0$. Hence

$$\frac{R}{R_N} = \frac{2 H_0}{H_k(H_0) + H_{c2}}.$$  

(59)
By using the calculated value of $H_k(H_0)$ we are able to construct a theoretical curve for $R/R_N$ as a function of $H_0$. This theoretical curve together with the measured one are plotted in fig. 8. The agreement is very good.

![Graph showing experimental and calculated curves of $R/R_N$ as a function of $H_0$.]

Fig. 8. Experimental and calculated curves of $R/R_N$ as a function of the applied field $H_0$.

5. Remarks

5.1. Validity of the model

We have used the picture of undisturbed cylindrical symmetric vortices, and in addition we have used London's equation for the superconducting region, which means that we have treated $n_0$ as a constant in this region. A consequence of this is that the validity of the calculations is restricted to values of $H_0$ which are rather less than $H_{c2}$. The result for the flow resistivity, however, seems to indicate that the agreement with the experiment is quite acceptable even in the region where $H_0$ approaches $H_{c2}$. We did not consider dissipation of energy outside the core which means that we restricted ourselves to temperatures far below the critical temperature $T_c$.

5.2. Type-I superconductors

There is some experimental evidence for longitudinal and transverse voltages on type-I superconductors in the intermediate state 6). These voltages can be explained by assuming moving normal spots instead of Abrikosov vortices. It is tempting to think that this situation can be described by the model given above with some slight modifications. Then we have normal regions containing many flux quanta instead of one flux quantum and $H_k$ can be replaced by the thermo-
dynamic critical field $H_0$ which simplifies the dependence of the voltages across the specimen on $H_0$. However, one cannot then rely on London’s equation, and the Pippard relation between current and field has to be used, which complicates the calculations considerably.

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