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BOOLEAN DIFFERENTIAL CALCULUS

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Abstract

After a brief outline of classical concepts relative to Boolean differential calculus, a theoretical study of various differential operators is undertaken. Application of these concepts to several important problems arising in switching practice is mentioned.

General notations

The following notations are used in this paper:

Boolean disjunction : $+$ or $\sum$

Boolean conjunction : no symbol or $\prod$

Boolean negation : $'$

Modulo two sum : $\oplus$ or $\sum$ (modulo two)

Boolean exponentiation: $x^{(e)} = x'$ if $e = 0$

$x^{(e)} = x$ if $e = 1$

$\mathbf{x}^{(\mathbf{e})} = x_0^{(e_0)}\, x_1^{(e_1)} \ldots x_{n-1}^{(e_{n-1})}$

$x^e = 1$ if $e = 0$

$x^e = x$ if $e = 1$

$\mathbf{x}^{\mathbf{e}} = x_0^{e_0}\, x_1^{e_1} \ldots x_{n-1}^{e_{n-1}}$

If D is an operator : $D\mathbf{x} = Dx_0, Dx_1, \ldots, Dx_{n-1}$

$D\mathbf{x}^{\mathbf{e}} = Dx_0^{e_0}\, x_1^{e_1} \ldots x_{n-1}^{e_{n-1}}$

$(D\mathbf{x})^{(\mathbf{e})} = (Dx_0)^{(e_0)}\, (Dx_1)^{(e_1)} \ldots (Dx_{n-1})^{(e_{n-1})}$

$(D\mathbf{x})^{\mathbf{e}} = (Dx_0)^{e_0}\, (Dx_1)^{e_1} \ldots (Dx_{n-1})^{e_{n-1}}$

$f_{\mathbf{x}_1=\mathbf{a}_1}$ or $f(\mathbf{x}_1 = \mathbf{a}_1)$ is the value of f for $\mathbf{x}_1 = \mathbf{a}_1$

$f(\mathbf{x}_1')$ is the value of f for $\mathbf{x}_1$ substituted by $\mathbf{x}_1'$ where this last notation means the vector of complemented literals.

1. Introduction

The concept of a differential of a Boolean function was introduced in a previous paper [6]). It was shown how the transient behaviour of a Boolean function is completely characterized by the algebraic properties of its associated differential.

This paper is mainly concerned with a further analysis of the variational properties of Boolean functions. It will be shown how a slight modification of the definition of the differential allows to solve a much larger class of problems without modifying the algorithms and procedures with the earlier definition of the differential. A large variety of problems, such as hazard detection, fault diagnosis, decomposition of Boolean functions and research of the prime implicants, can all be solved by only using the concept of differential of a Boolean function, and this discloses some new conceptual affinities between these various problems.

Akers [1]) defined the concept of partial derivative of a Boolean function and showed that it is a measure of the invariance of the function on an edge of the 2^n cube. Further he proved that this concept is closely related to the classical concept of partial derivative and obtained important theorems analogous to McLaurin and Taylor expansions, giving thus a basis to Boolean calculus. In sec. 2 the concept of a partial differential is defined and used to examine various formal properties of Boolean functions. Our concept is in fact a parametric form of the partial derivative of Akers.

The concept of sensitivity function was also introduced by Akers and later developed by others [2,3,4]). In particular, this concept was presented by Sellers and Bearnson as a natural extension of the partial derivative and was accordingly called multiple partial derivative. The sensitivity function is a measure of the invariance of a Boolean function on a diagonal of a sub-cube. Its importance arises mainly from the fact that it is used to analyze the effect of multiple errors on the outputs of logic circuits. The concept of total differential of a function is defined in sec. 3: it is a parametric form of the set of all the sensitivity functions associated with a Boolean function.

Finally Akers also introduced the Δ-operator which is a measure of the invariance of a Boolean function on a sub-cube. One of the primary advantages of using this concept is that it allows to formalize the theory of decomposition of Boolean functions of Curtis [5]) and Ashenhurst [11]). The total variation of a function is defined in sec. 4: it is a parametric form of the set of all the Δ-operators associated with a Boolean function.

Section 5 is devoted to a review of some possible applications of the mathematical concepts previously introduced.

2. The partial differential

Given a function f of n variables $x_0, x_1, \ldots, x_{n-1} = \mathbf{x}$ the *partial derivative* of f with respect to a variable $x_i \in \mathbf{x}$ will be denoted $\partial f/\partial x_i$ and is defined by Akers as follows.

Definition 1

$$\frac{\partial f}{\partial x_i} = f(x_i) \oplus f(x_i'). \tag{1}$$

If $\mathbf{x}_1, \mathbf{x}_2$ is a partition of $\mathbf{x}$, the *multiple partial derivative* of the function f with respect to the p variables in $\mathbf{x}_1$ will be denoted $\partial f/\partial \mathbf{x}_1$ and is defined as follows.

Definition 2

$$\frac{\partial f}{\partial \mathbf{x}_1} = \frac{\partial}{\partial x_0}\left(\frac{\partial}{\partial x_1}\left(\cdots\frac{\partial f}{\partial x_{p-1}}\right)\cdots\right), \quad \mathbf{x}_1 = (x_0, x_1, \ldots, x_{p-1}). \tag{2}$$

Most of their properties were derived in a straightforward manner in a paper by Akers [1]). The most important property of the partial derivative is that it is equal to 1 when the logic values of f are different for direct and complemented values of variable x_i, and equal to 0 if the logic value of f is the same for both direct and complemented values of variable x_i. It follows that the partial derivative of a Boolean function is a measure of the invariance of this function on an edge of the cube which is the domain of definition of this function.

The following two theorems, due to Akers, show the importance of the concept of partial derivative as a theoretical tool by giving the relations of these derivatives with the coefficients of the well-known ring expansion of Boolean functions.

Theorem 1

Any Boolean function $f(\mathbf{x}_1, \mathbf{x}_2)$ may be expanded as

$$f(\mathbf{x}_1, \mathbf{x}_2) = \sum_{\mathbf{e}} \left(\frac{\partial f}{\partial {\mathbf{x}_1}^{\mathbf{e}}}\right)_{\mathbf{x}_1 = \mathbf{0}} {\mathbf{x}_1}^{\mathbf{e}}, \quad 0 \leqslant \mathbf{e} \leqslant 2^p - 1. \tag{3}$$

Theorem 1 is called by Akers the *Mac Laurin expansion of a Boolean function,* since it involves only the values of the partial derivatives at $\mathbf{x}_1 = \mathbf{0}$. The theorem 2 will analogously be called the *Taylor expansion of the Boolean function.*

Theorem 2

For any Boolean function $f(\mathbf{x}_1, \mathbf{x}_2)$:

$$f(\mathbf{x}_1, \mathbf{x}_2) = \sum_{\mathbf{e}} \left(\frac{\partial f}{\partial {\mathbf{x}_1}^{\mathbf{e}}}\right)_{\mathbf{x}_1 = \mathbf{h}_1} (\mathbf{x}_1 \oplus \mathbf{h}_1)^{\mathbf{e}}, \quad 0 \leqslant \mathbf{e} \leqslant 2^p - 1. \tag{4}$$

Some other expansions arise from searching to know if a Boolean function is completely determined by means of the set of all its simple derivatives. An answer to this question is given by means of the following theorem.

Theorem 3

A Boolean function $f(\mathbf{x})$ is determined within an arbitrary binary constant when all its simple derivatives are known.
Proof. Definition 1 allows us to write:

$$f(\mathbf{x}) = f(x_0, \ldots, x_{n-2}, h_{n-1}) \oplus x_{n-1}^{h_{n-1}'} \frac{\partial f}{\partial x_{n-1}}. \tag{5}$$

By expanding in the same way the first term of the right-hand side of this expression, and by iterating the process one obtains:

$$f(\mathbf{x}) = f(\mathbf{h}) \oplus \sum_{i=0}^{n-1} x_i^{h_i'} \left(\frac{\partial f}{\partial x_i}\right)_{x_j = h_j}, \quad j > i. \tag{6}$$

The proof being of the constructive type, one obtains an expansion of $f(\mathbf{x})$ in terms of its simple derivatives.

The expansion (7) has been derived by Davio and Piret [4]) and constitutes a corollary of theorem 2:

$$f(\mathbf{x}) \oplus f(\mathbf{x} \oplus \mathbf{h}) = \sum_{\mathbf{e}} \left(\frac{\partial f}{\partial \mathbf{x}^{\mathbf{e}}}\right) \mathbf{h}^{\mathbf{e}}, \quad 0 < \mathbf{e} \leqslant 2^n - 1. \tag{7}$$

Let us now introduce the concept of *partial differential*; the *differential* of Boolean variables and functions has been introduced by the author in ref. 6. More-complete and more-general definitions will be given in this paper.

Definition 3

The *differential* $\mathrm{d}x_i$ of the variable x_i is the increment of x_i.
Let us recall that the increment in a two-valued Boolean algebra is obtained by performing the exclusive-OR operation; the differential of a Boolean variable is thus itself a Boolean variable.

Definition 4

The *partial differential* $\mathrm{d}_{x_i} f$ of the function f with respect to a variable x_i is the increment of f due to the increment of x_i.

This definition leads to the following relation:

$$\begin{aligned} \mathrm{d}_{x_i} f &= f(x_i) \oplus f(x_i \oplus \mathrm{d}x_i), \\ &= \frac{\partial f}{\partial x_i} \mathrm{d}x_i. \end{aligned} \tag{8}$$

The partial differential is thus a parametric form of the partial derivative, the parameter being the differential $\mathrm{d}x_i$, and consequently enjoys similar properties.

Definition 5

The multiple partial differential $\mathrm{d}_{\mathbf{x}_1} f$ of f with respect to the p variables in $\mathbf{x}_1$ is the function

$$\begin{aligned} \mathrm{d}_{\mathbf{x}_1} f &= \mathrm{d}_{x_0} (\mathrm{d}_{x_1} (\ldots \mathrm{d}_{x_{p-1}} f) \ldots), \\ &= \frac{\partial f}{\partial \mathbf{x}_1} \prod \mathrm{d}x_i, \quad x_i \in \mathbf{x}_1 \doteq x_0, x_1, \ldots, x_{p-1}. \end{aligned} \tag{9}$$

3. The total differential

3.1. *Definitions and basic properties*

Definition 6

The *total differential* df of a function $f(\mathbf{x})$ is the increment of this function due to the increment $d\mathbf{x}$ of $\mathbf{x}$.
This definition leads to the following relation:

$$df = f(\mathbf{x}) \oplus f(\mathbf{x} \oplus d\mathbf{x}). \tag{10}$$

From (10) one deduces:

$$(df)_{d\mathbf{x}_1 = 1,\, d\mathbf{x}_2 = 0} = f(\mathbf{x}_1, \mathbf{x}_2) \oplus f(\mathbf{x}_1', \mathbf{x}_2). \tag{11}$$

But this last expression is by definition the *sensitivity function* $Sf/S\mathbf{x}_1$ of the function f with respect to the p variables in $\mathbf{x}_1$. The importance of the sensitivity function was suggested in a paper of Sellers, Hsiao and Bearnson [2]; its main properties were studied in a recent paper of Davio and Piret [4]. From expression (11) it appears that the sensitivity function is equal to 1 when the logic values of f are different for direct and complemented values of binary vector $\mathbf{x}_1$, and equal to 0 if the logic value of f is the same for both direct and complemented values of vector $\mathbf{x}_1$. The sensitivity function is thus a measure of the invariance of a Boolean function on a diagonal of a sub-cube. From (11) one deduces that *the total differential of a function is a parametric form of the set of all the sensitivity functions associated with a given Boolean function.*

By making in eq. (7) $\mathbf{h}$ equal to $d\mathbf{x}$, one obtains in view of (9) and (10) the two following expressions for the total differential:

$$df = \sum_{\mathbf{e}} \frac{\partial f}{\partial \mathbf{x}^{\mathbf{e}}} (d\mathbf{x})^{\mathbf{e}}, \quad 0 < \mathbf{e} \leqslant 2^n - 1, \tag{12}$$

$$= \sum_{\mathbf{e}} d_{\mathbf{x}^{\mathbf{e}}} f, \quad 0 < \mathbf{e} \leqslant 2^n - 1. \tag{13}$$

Let us examine a few of the basic properties of the total differential operator d, indicating the method of proof. If $f(\mathbf{x}, \mathbf{y})$ is a function of the variables $x_i \in \mathbf{x}$ and $y_j \in \mathbf{y}$, the latter being themselves functions of the x_i's, one has

$$\begin{aligned} df[\mathbf{x}, \mathbf{y}(\mathbf{x})] &= f[\mathbf{x}, \mathbf{y}(\mathbf{x})] \oplus f[\mathbf{x} \oplus d\mathbf{x}, \mathbf{y}(\mathbf{x} \oplus d\mathbf{x})], \\ &= f[\mathbf{x}, \mathbf{y}(\mathbf{x})] \oplus f[\mathbf{x} \oplus d\mathbf{x}, \mathbf{y}(\mathbf{x}) \oplus d\mathbf{y}]. \end{aligned} \tag{14}$$

It follows that *the formal expression of the total differential of a function is the same, whether the variables are independent or not.* This principle is known in classical calculus as that of the *invariance of the total differential.* Elementary computations show us that the total differential operator satisfies properties P_1 to P_7 below (a = constant):

$$P_1: \quad da = 0, \tag{15}$$

$$P_2: \quad d(af) = a\,df, \tag{16}$$

$$P_3: \quad d(f') = df. \tag{17}$$

These properties immediately follow from correspondent properties relative to partial derivatives.

$$P_4: \quad d(df) = 0. \tag{18}$$

If a multiple partial derivative with respect to p variables appears in the expression of $d(df)$, then it can easily be verified that this derivative necessarily appears $2^p - 2$ times; this number being even, the corresponding derivative disappears.

$$P_5: \quad d(fg) = f\,dg \oplus g\,df \oplus df\,dg, \tag{19}$$

$$P_6: \quad d(f+g) = f'\,dg \oplus g'\,df \oplus df\,dg, \tag{20}$$

$$P_7: \quad d(f \oplus g) = df \oplus dg. \tag{21}$$

These last properties are obtained by differentiating a function $F(fg)$ which is successively made equal to fg, $f+g$ and $f \oplus g$ and by taking into account the invariance principle of the total differential. Evidently the sensitivity operator S/Sx_1 satisfies also properties P_1 to P_7.

The invariance principle and the properties P_1 and P_7 lead to the following principle.

Given a Boolean equation between a set of variables, independent or not, it is always allowed to make equal the total differentials of its two sides. This last operation will be called: *to differentiate totally a Boolean equation.* The total differential may also be expressed as a function of all the sensitivity functions, that is

$$df = \sum_{e} \frac{Sf}{Sx^e}(dx)^{(e)}, \quad 0 < e \leqslant 2^n - 1. \tag{22}$$

The equivalence between expressions (12) and (22) leads to the following relation between sensitivity functions and partial derivatives:

$$\frac{Sf}{Sx_1} = \sum_{e} \frac{\partial f}{\partial x_1{}^e}, \quad 0 < e \leqslant 2^p - 1. \tag{23}$$

Let us finally note that an interchange of the roles played in (7) by $\mathbf{x}$ and $\mathbf{h}$ allows us to write:

$$\mathrm{d}f=\sum_{\mathbf{e}}\left(\frac{\partial f}{\partial \mathbf{x}^{\mathbf{e}}}\right)_{\mathbf{x}=\mathrm{d}\mathbf{x}}\mathbf{x}^{\mathbf{e}},\quad 0\leqslant \mathbf{e}<2^{n}-1, \tag{24}$$

which gives us a last formulation of the total differential.

3.2. *Differential operators of functions of functions*

Let us consider a function $f[\mathbf{x}, \mathbf{y}(\mathbf{x})]$, where $\mathbf{x}$ and $\mathbf{y}$ are n- and m-dimensional vectors respectively; the *total derivative* of f with respect to a variable x_i is denoted $\mathrm{d}f/\mathrm{d}x_i$ and is defined as follows:

Definition 7

$$\frac{\mathrm{d}f}{\mathrm{d}x_i}=f[x_i, \mathbf{y}(x_i)]\oplus f[x_i', \mathbf{y}(x_i')]. \tag{25}$$

If $\mathbf{x}_1, \mathbf{x}_2$ is a partition of $\mathbf{x}$, the *multiple total derivative* of the function f with respect to the p variables in $\mathbf{x}_1$ will be denoted $\mathrm{d}f/\mathrm{d}\mathbf{x}_1$ and is defined as follows.

Definition 8

$$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}_1}=\frac{\mathrm{d}}{\mathrm{d}x_0}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\cdots\frac{\mathrm{d}f}{\mathrm{d}x_{p-1}}\right)\cdots\right),\quad \mathbf{x}_1=(x_0, x_1, \ldots, x_{p-1}). \tag{26}$$

The *total sensitivity function* $Sf/S\mathbf{x}_1$ of the function f with respect to the p variables in $\mathbf{x}_1$ is the following function.

Definition 9

$$\frac{Sf}{S\mathbf{x}_1}=f[\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}(\mathbf{x}_1, \mathbf{x}_2)]\oplus f[\mathbf{x}_1', \mathbf{x}_2, \mathbf{y}(\mathbf{x}_1', \mathbf{x}_2)]. \tag{27}$$

Let us consider a function $\phi(\mathbf{x})$ such that

$$f[\mathbf{x}, \mathbf{y}(\mathbf{x})]=\phi(\mathbf{x}). \tag{28}$$

Since relation (28) holds for any value of the vector $\mathbf{x}$, the following identities appear:

$$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}_1}=\frac{\partial\phi}{\partial\mathbf{x}_1}, \tag{29}$$

$$\frac{Sf}{S\mathbf{x}_1}=\frac{S\phi}{S\mathbf{x}_1}, \tag{30}$$

$$\mathrm{d}f=\mathrm{d}\phi. \tag{31}$$

This yields the following image principle:
If any equality is true when involving the operators $\partial/\partial x_1$, S/Sx_1, d *and the function* ϕ, *so is its image when interchanging* $\partial/\partial x_1$, S/Sx_1, ϕ *and* d/dx_1, S/Sx_1, f *and when leaving* d *unchanged.*

The author has derived some computational formulas in order to obtain the total operators defined hereabove in terms of the partial operators. For example, one has

$$\frac{df}{dx_i} = \frac{\partial f}{\partial x_i} \oplus \sum_{e} \left(\frac{\partial f}{\partial \mathbf{y}^{e}} \oplus \frac{\partial f}{\partial (x_i\, \mathbf{y}^{e})} \right) \left(\frac{\partial \mathbf{y}}{\partial x_i} \right)^{e}, \quad 0 < \mathbf{e} \leqslant 2^{m} - 1, \tag{32}$$

with

$$\frac{\partial \mathbf{y}}{\partial x_i} = \left(\frac{\partial y_0}{\partial x_i}, \frac{\partial y_1}{\partial x_i}, \ldots, \frac{\partial y_{m-1}}{\partial x_i} \right). \tag{33}$$

4. The total variation

4.1. *Definitions and basic properties*

Definition 10

The *variation* δx_i of an independent variable is the increment of x_i.
It follows that

$$\delta x_i = dx_i. \tag{34}$$

Definition 11

The *total variation* δf of a function $f(\mathbf{x})$ is the maximal increment of this function due to the total variation $\delta \mathbf{x}$ of $\mathbf{x}$.
Let us first consider a function of independent variables only, that is $f = f(\mathbf{x})$; definitions (10) and (11) lead to the following relations:

$$\delta f = \sum_{\mathbf{e}_1} [f(\mathbf{x}) \oplus f(x \oplus \mathbf{e}_1\, \delta \mathbf{x})], \quad 0 < \mathbf{e}_1 \leqslant 2^{n} - 1,$$

$$= \sum_{\mathbf{e}_1} \sum_{e} \left(\frac{\partial f}{\partial \mathbf{x}^{e}} (\mathbf{e}_1\, \delta \mathbf{x})^{e} \right), \quad 0 < \mathbf{e} \leqslant 2^{n} - 1,$$

$$= \sum_{e} \left(\frac{\partial f}{\partial \mathbf{x}^{e}} \right) (\delta \mathbf{x})^{e}, \tag{35}$$

$$= \sum_{e} d_{\mathbf{x}^{e}} f. \tag{36}$$

From (35) one deduces:

$$(\delta f)_{\delta x_1=1,\, \delta x_2=0} = \sum_{e} \frac{\delta f}{\delta x_1{}^e}, \quad 0 < e \leqslant 2^p - 1. \tag{37}$$

But this last expression is by definition *the Δ-operator $\Delta f/\Delta x_1$* of the function f with respect to the p variables in x_1. The importance of the Δ-operator was suggested in the paper of Akers [1]); he proved the two following theorems.

Theorem 4

$\Delta f(x_1, x_2)/\Delta x_1$ is independent of x_1.

Theorem 5

$\Delta f(x)/\Delta x$ is 0 or 1 and is 0 if and only if f is itself 0 or 1.

The Δ-operator is thus a measure of the invariance of a Boolean function on a sub-cube. From (37) one deduces that *the total variation is a parametric form of the set of all the Δ-operators associated with a given Boolean function.*

Let us examine a few of the basic properties of the total variation operator δ. If $f = f[x, y(x)]$, definition 11 leads directly to the following expression:

$$\delta f = \sum_{e} \frac{\delta f}{\delta (x\, y)^e} [\delta(x, y)]^e, \quad 0 < e \leqslant 2^{m+n} - 1, \tag{38}$$

which shows us that the invariance principle also holds for the total variation when this last is expressed by means of the total variations of its variables. Evidently, expression (36) does not satisfy this last condition and thus only holds for functions of independent variables.

Elementary computations show us that the total variation operator satisfies properties P_1 to P_2 while properties P_5 to P_7 must be substituted by the following ones:

$$P_{5b}: \quad \delta(fg) = f\,\delta g + g\,\delta f + \delta f\,\delta g, \tag{39}$$

$$P_{6b}: \quad \delta(f + g) = f'\,\delta g + g'\,\delta f + \delta f\,\delta g, \tag{40}$$

$$P_{7b}: \quad \delta(f \oplus g) = \delta f + \delta g. \tag{41}$$

It must be pointed out that the total variation of a function depends on the function expression, that is on the possible y_j's present in its formulation. In particular a function $f(x, y)$ the total variation of which is not zero may derive from a function $\phi(x)$ which is identically zero. More precisely, for any functions $\phi(x)$ and $f[x, y(x)]$ that are such that $\phi(x) = f[x, y(x)]$, one has:

$$d\phi = df \leqslant \delta\phi \leqslant \delta f. \tag{42}$$

From (38) one deduces that the total Δ-operator defined below satisfies properties P_1 to P_4 and P_{5b} to P_{7b}. Finally the total variation of a function of independent variables may also be expressed as a function of all the Δ-operators, that is

$$\delta f = \sum_{\mathbf{e}} \left(\frac{\Delta f}{\Delta \mathbf{x}^{\mathbf{e}}}\right) (\delta \mathbf{x})^{(\mathbf{e})}, \quad 0 < \mathbf{e} \leqslant 2^n - 1. \tag{43}$$

4.2. *Variational operators of functions of functions*

The *partial variation* of $f = f[\mathbf{x}, \mathbf{y}(\mathbf{x})]$ due to the variation of $x_i \in \mathbf{x}$ will be denoted $\delta f/\delta x_i$ and is defined by means of the following relation:

Definition 12

$$\frac{\delta f}{\delta x_i} = \sum_{\mathbf{e}_i} \{f(\mathbf{x}, \mathbf{y}) \oplus f[x_i \oplus e_{m_i}, y_j(x_i \oplus e_{j_i})]\},$$
$$j = 0, 1, \ldots, m-1; \quad \mathbf{e}_i = e_{0_i}, \ldots, e_{m_i}; \quad 0 < \mathbf{e}_i \leqslant 2^{m+1} - 1. \tag{44}$$

By expanding the right-hand side of (44) one obtains in view of (7):

$$\frac{\delta f}{\delta x_i} = \sum_{\mathbf{e}} \frac{\partial f}{\partial (x_i \mathbf{y})^{\mathbf{e}}} \left[1, \frac{\partial \mathbf{y}}{\partial x_i}\right]^{\mathbf{e}}, \quad 0 < \mathbf{e} \leqslant 2^{m+1} - 1,$$
$$= \frac{\partial f}{\partial x_i} + \sum_{\mathbf{e}} \left[\frac{\partial f}{\partial \mathbf{y}^{\mathbf{e}}} + \frac{\partial f}{\partial (x_i \mathbf{y}^{\mathbf{e}})}\right] \left(\frac{\partial \mathbf{y}}{\partial x_i}\right)^{\mathbf{e}}, \quad 0 < \mathbf{e} \leqslant 2^m - 1. \tag{45}$$

The *total Δ-operator* of $f = f[\mathbf{x}, \mathbf{y}(\mathbf{x})]$ with respect to the p variables in $\mathbf{x}_1$ will be denoted $Df/D\mathbf{x}_1$ and is defined by means of the following relation.

Definition 13

$$\frac{Df}{D\mathbf{x}_1} = \sum_{\mathbf{e}_0, \ldots, \mathbf{e}_m} \{f(\mathbf{x}, \mathbf{y}) \oplus f[\mathbf{x}_1 \oplus \mathbf{e}_m, y_j(\mathbf{x}_1 \oplus \mathbf{e}_j)]\},$$
$$j = 0, 1, \ldots, m-1; \quad 0 \leqslant \mathbf{e}_0, \mathbf{e}_1, \ldots, \leqslant 2^p - 1; \quad 0 < \sum_i \mathbf{e}_i. \tag{46}$$

By expanding the right-hand side of (46) one obtains in view of (7):

$$\frac{Df}{D\mathbf{x}_1} = \sum_{\mathbf{e}} \frac{\partial f}{\partial (\mathbf{x}_1 \mathbf{y})^{\mathbf{e}}} \left[\mathbf{1}_p, \frac{\Delta \mathbf{y}}{\Delta \mathbf{x}_1}\right]^{\mathbf{e}}, \qquad 0 < \mathbf{e} \leqslant 2^{p+m} - 1,$$
$$= \frac{\Delta f}{\Delta \mathbf{x}_1} + \sum_{\mathbf{e}_1 \mathbf{e}_2} \frac{\partial f}{\partial (\mathbf{x}_1^{\mathbf{e}_1} \mathbf{y}^{\mathbf{e}_2})} \left(\frac{\Delta \mathbf{y}}{\Delta \mathbf{x}_1}\right)^{\mathbf{e}_2}, \quad 0 \leqslant \mathbf{e}_1 \leqslant 2^p - 1, \tag{47}$$
$$0 < \mathbf{e}_2 \leqslant 2^m - 1,$$

with

$$\frac{\Delta \mathbf{y}}{\Delta \mathbf{x}_1} = \left(\frac{\Delta y_0}{\Delta \mathbf{x}_1}, \frac{\Delta y_1}{\Delta \mathbf{x}_1}, \ldots, \frac{\Delta y_{m-1}}{\Delta \mathbf{x}_1} \right). \tag{48}$$

As it can easily be verified the total variation operator is a parametric form of the set of operators $D/D\mathbf{x}^c$.

4.3. *Reduced variational operators*

Definition 14

A *reduced variational operator* associated with a variational operator is that operator where all terms containing multiple derivatives were dropped.
The *reduced total variation* of function f will be denoted $\delta^R f$; let us first observe that independent variables are such that

$$\mathrm{d}x_i = \delta x_i = \delta^R x_i. \tag{49}$$

If $f = f(\mathbf{x})$, one has

$$\delta^R f = \sum_i \frac{\partial f}{\partial x_i} \delta^R x_i. \tag{50}$$

If $f = f[\mathbf{x}, \mathbf{y}(\mathbf{x})]$ the following relations hold:

$$\delta^R f = \sum_i \frac{\partial f}{\partial x_i} \delta^R x_i + \sum_j \frac{\partial f}{\partial y_j} \delta^R y_j, \tag{51}$$

$$\delta^R y_j = \sum_i \frac{\partial y_j}{\partial x_i} \delta^R x_i, \tag{52}$$

and consecutively, one has

$$\delta^R f = \sum_i \left(\frac{\partial f}{\partial x_i} + \sum_j \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right) \delta^R x_i, \tag{53a}$$

$$= \sum_i \frac{\delta^R f}{\delta^R x_i} \delta^R x_i. \tag{53b}$$

Relations (53a) and (53b) provide us with a formal definition of the *reduced partial variation* with respect to the variation of one variable which is denoted $\delta^R/\delta^R x_i$. The importance of the reduced variation arises from the following theorem.

Theorem 6

The total variation of a function is identically zero if and only if the reduced variation is identically zero.
Proof. Since $\delta f \geqslant \delta^R f$, $\delta f \equiv 0$ implies $\delta^R f \equiv 0$.
If $\delta^R f \equiv 0$, then $\partial f/\partial x_i = 0$ for each x_i and $\partial f/\partial(x_i\ \mathbf{x})^{\mathbf{e}} = 0$ for each x_i and each $\mathbf{e}$; this proves the theorem.

It must be noted that theorem 6 strictly holds only for functions of independent variables. When one deals with functions of the form $f(\mathbf{x}, \mathbf{y})$, theorem 6 remains true if the y_j's are momentarily considered as independent variables, that is, if they are not substituted by means of their expression as functions of the x_i's. This question has been extensively treated in ref. 6 where the *reduced total variation* was called *differential* since no confusion with other operators could occur.

5. Applications

5.1. *Prime implicants and prime implicates of a Boolean function*

The knowledge of the prime implicants and of the prime implicates of a Boolean function is requested for obtaining the solution of many problems appearing in switching theory, as e.g. the simplification of switching functions, the design of minimal logic circuits, the hazard-free design of logic circuits and the synthesis of multi-level logic networks. Implicants and implicates of a Boolean function being sub-cubes where this function takes the constant values 1 and 0 respectively, that is where the total variation degenerates, it is obvious that these concepts are strongly connected with those of the various differential operators quoted hereabove. Algebraic relations have been stated by the author which relate the classical concepts of prime implicants and of prime implicates of a Boolean function to the various differential operators which were defined in this paper. A further paper will be devoted to this question.

5.2. *Transient analysis of logical networks*

One of the more classical problems relative to transient analysis of binary switching networks is the *detection of hazards*. Whenever the input signals of a combinational or sequential switching network are changed, i.e. for transient conditions, the use of the truth table associated with the considered network and which was built by using the properties of two-valued Boolean algebra can lead to an incorrect prediction of the real behaviour of the network; the network is then said to contain a hazard for that input change. The specific reason for this incorrect prediction is that two-valued Boolean algebra implies the assumption that the propagation time of signals in the different parts of the network is strictly zero. This assumption does no longer hold when dealing with physical networks.

Let us consider a vector **x** and let us define as follows two fixed values of this vector:

$$\begin{aligned}\mathbf{a} &= (a_0, \ldots, a_{p-1}, a_p, \ldots, a_{n-1}) = (\mathbf{a}_1, \mathbf{a}_2),\\ \mathbf{b} &= (a_0', \ldots, a_{p-1}', a_p, \ldots, a_{n-1}) = (\mathbf{a}_1', \mathbf{a}_2).\end{aligned} \tag{54}$$

The *transition sub-cube* (**a**/**b**) is defined as the set of 2^p different values which may be taken by the vector $(\mathbf{x}_1, \mathbf{a}_2)$.

Let us consider a network realizing the function $f(\mathbf{x})$. The propagation delays inherently associated with the inputs of this network make that, when considering a transition from input state **a** to input state **b**, each vertex of the transition sub-cube (**a**/**b**) may be reached instead of only the two extremal vertices, that is **a** and **b**. When this causes a malfunction on the network output, a *function hazard* is said to exist for that transition.

Let us consider a vector $[\mathbf{x}, \mathbf{y}(\mathbf{x})]$; the *dynamic transition sub-cube* $[\mathbf{a}, \mathbf{y}(\mathbf{a})/\mathbf{b}, \mathbf{y}(\mathbf{b})]$ is defined as the set of $2^{p(m+1)}$ different values which may be taken by the vector

$$[(\mathbf{x}_1^{(\mathbf{e}_m)}, \mathbf{a}_2), y_0(\mathbf{x}_1^{(\mathbf{e}_0)}, \mathbf{a}_2), \ldots, y_{m-1}(\mathbf{x}_1^{(\mathbf{e}_{m-1})}, \mathbf{a}_2)],$$

where $0 \leqslant \mathbf{e}_0, \ldots, \mathbf{e}_m \leqslant 2^p - 1$. Evidently, the dynamic transition sub-cube reduces to the transition sub-cube if **y** is an empty vector.

Let us consider a network realizing the function $f(\mathbf{x})$. The propagation delays inherently associated with the gates and with the wires interconnecting these gates of the network make that the function $f(\mathbf{x})$ must be replaced by a function $F[\mathbf{x}, \mathbf{y}(\mathbf{x})] = f(\mathbf{x})$. How to obtain this function was explained in ref. 6; the new degrees of freedom brought into the network by the internal delays make that each vertex of the dynamic transition sub-cube defined before may now be reached when considering an input change from **a** to **b**. When this causes a malfunction on the network output, a *logic hazard* is said to exist for that transition.

In order to make clear the relations between the total variation of a function and the transient behaviour of a logical network, let us consider the following theorem, the proof of which can be given immediately.

Theorem 7

The total variation δF of a function $F[\mathbf{x}, \mathbf{y}(\mathbf{x})]$ is the maximal increment of this function between two arbitrary vertices of the dynamic transition sub-cube produced by the increment $\delta\mathbf{x}$ of the vector **x** of independent variables.

The complete terminology connected with the various kinds of combinatorial hazards has been extensively studied in ref. 6 and will not be recalled here. Let us consider the following function:

$$T_{fs}(f) = \mathrm{d}f(\mathbf{x}) \oplus \delta f(\mathbf{x}). \tag{55}$$

Theorem 8

The function T_{fs} indicates all the transitions for which a function static hazard occurs.

Example

Consider the function

$$f = x_1 x_3 + x_2 x_3'. \tag{56}$$

One has

$$df \oplus \delta f = (x_1 \oplus x_2 \oplus x_3)\, dx_1 (dx_2)' dx_3 + (x_1 \oplus x_2 \oplus x_3')(dx_1)' dx_2 dx_3 + \\ + (x_1 \oplus x_2)\, dx_1 dx_2 dx_3. \tag{57}$$

The transitions containing a function static hazard are thus characterized by (a_i = fixed value of x_i):

$$\begin{aligned} &dx_1 = dx_3 = 1, \quad dx_2 = 0, \quad a_1 \oplus a_2 \oplus a_3 = 1, \\ &dx_2 = dx_3 = 1, \quad dx_1 = 0, \quad a_1 \oplus a_2 \oplus a_3' = 1, \\ &dx_1 = dx_2 = dx_3 = 1, \quad a_1 \oplus a_2 = 1. \end{aligned} \tag{58}$$

It has been proven in ref. 6 that all logic static hazards on the output of a combinatorial network could be detected providing a good choice of a function $F[\mathrm{x}, \mathrm{y}(\mathrm{x})] = f(\mathrm{x})$, where f is the binary function to be realized by the network. How to obtain this function was also shown in ref. 6. Let us consider the following functions:

$$\begin{aligned} T_{ls}(f, F) &= \delta f \oplus \delta F, \\ T_{fls}(f, F) &= df \oplus \delta F. \end{aligned} \tag{59}$$

The following theorems hold in view of theorem 7.

Theorem 9

The function $T_{ls}(f, F)$ indicates all the transitions for which a logic static hazard occurs.

Theorem 10

The function $T_{fls}(f, F)$ indicates all the transitions for which a static (function or logic) hazard occurs.

Example

Consider the two-level realization of $f = x_1 x_3 + x_2 x_3'$ by means of an

AND–OR network where the two AND-gates realize the functions $x_1 x_3$ and $x_2 x_3'$ respectively. The function $F(\mathbf{x}, \mathbf{y})$ to be considered is [6].

$$F = x_1 x_3 + y, \qquad y = x_2 x_3'. \tag{60}$$

One has

$$T_{ls}(f, F) = x_1 x_2 (\mathrm{d}x_1)' (\mathrm{d}x_2)' \mathrm{d}x_3. \tag{61}$$

The only transition containing a logic static hazard is characterized by

$$\mathrm{d}x_1 = \mathrm{d}x_2 = 0, \quad \mathrm{d}x_3 = 1, \quad x_1 = x_2 = 1. \tag{62}$$

5.3. *Error detection of combinatorial logic circuits*

An *error* on a logic input x_i or on an internal connection y_j is generally assumed to be either one of the following two types:

(1) *stuck type*: stuck at the binary value 1 or stuck at the binary value 0;

(2) *inversion type*: where the signal value is the inverse of the correct value.

Let us consider e.g. errors on an input x_i. Whether these errors will cause the output function $f(\mathbf{x})$ to be wrong or not is dependent on the value of $\partial f/\partial x_i$. As a direct consequence of theorem 4, one has

(1) if $\partial f/\partial x_i = 0$, then an error in x_i does not cause an error in $f(\mathbf{x})$;

(2) if $\partial f/\partial x_i = g(\mathbf{x})$, then an error in x_i will cause an error in $f(\mathbf{x})$ if and only if $g(\mathbf{x}) = 1$;

(3) if $\partial f/\partial x_i = 1$, then an error in x_i will always cause an error in $f(\mathbf{x})$.

It is easy to see that the sensitivity function $\mathrm{S}f/\mathrm{S}\mathbf{x}_1$ must be considered for studying the effect of p simultaneous errors in the p inputs $\in \mathbf{x}_1$; similarly the Δ-operator $\Delta f/\Delta \mathbf{x}_1$ must be considered for studying the effect of an error in the inputs which constitute an arbitrary subset of $\mathbf{x}_1$. As a consequence the total differential $\mathrm{d}f$ and the total variation δf of f are adequate operators for studying the effects of an error in all the variables of a given set or in all the variables of an arbitrary subset of a given set respectively. Similarly the reduced variation $\delta^R f$ studies the effect of an error in one of the variables belonging to an arbitrary subset of $\mathbf{x}$.

Analogous considerations hold when considering errors in the internal connections of a network. One has then to compute the total differential, the total variations or the reduced variation of a function $F(\mathbf{x}, \mathbf{y})$ where each of the y_j's $\in \mathbf{y}$ corresponds to an internal connection of the network.

Example

Consider the two-level realization of $f = x_1 x_3 + x_2 x_3'$ by means of an AND–OR network where the two AND-gates realize the function $x_1 x_3$ and $x_2 x_3'$ respectively. One has

$$\mathrm{d}f = x_3 \,\mathrm{d}x_1 \oplus x_3' \,\mathrm{d}x_2 \oplus (x_1 \oplus x_2)\, \mathrm{d}x_3 \oplus \mathrm{d}x_1 \,\mathrm{d}x_3 \oplus \mathrm{d}x_2 \,\mathrm{d}x_3. \tag{63}$$

Let us e.g. test an error on x_1; a pair of test patterns is thus $x_3 = 1$, $dx_1 = 1$. Let us now test an error on the internal connection between the AND-gate realizing the function $x_1 x_3$ and the OR-gate. An auxiliary variable y will be associated with this connection. The following equations hold:

$$\begin{aligned} f &= y + x_2 x_3', \\ df &= x_3' y' dx_2 \oplus x_2 y' dx_3 \oplus (x_2' + x_3) dy \oplus y' dx_2 dx_3 \\ &\quad \oplus x_3' dx_2 dy \oplus x_2 dx_3 dy \oplus dx_2 dx_3 dy, \end{aligned} \tag{64}$$

$$\begin{aligned} y &= x_1 x_3, \\ dy &= x_3 dx_1 \oplus x_1 dx_3 \oplus dx_1 dx_3. \end{aligned} \tag{65}$$

In view of (64), a pair of test patterns for an error on y is thus $x_2 = 0$, $dy = 1$. This pair of test patterns becomes in view of (65) $x_2 = 0$, $x_3 = 1$, $dx_1 = 1$.

For further details concerning test-pattern generation, the reader will usefully refer to ref. 10.

5.4. *Decomposition of Boolean functions*

Assume it is desired to synthesize a switching combinatorial circuit whose output function is F. Two essential problems relative to the synthesis of F may be considered:

(1) Given a Boolean function $F(\mathbf{x})$ and m Boolean functions $y_j(\mathbf{x}_1)$, is it possible to obtain a Boolean function $F_1(\mathbf{y}, \mathbf{x}_2)$ such that $F = F_1$ where $(\mathbf{x}_1, \mathbf{x}_2)$ is a partition of $\mathbf{x}$?

(2) Given a Boolean function $F(\mathbf{x})$ and a Boolean function $F_1(\mathbf{y}, \mathbf{x}_2)$, is it possible to build p Boolean functions $y_j(\mathbf{x}_1)$ such that $F = F_1$?

These two problems can be solved at least formally, by solving the following Boolean equations:

$$\frac{\eth F}{\eth x_i} = \frac{dF_1}{dx_i} = \phi_i \left(\frac{\eth F_1}{\eth (x_i y^e)}, \frac{\eth y_j}{\eth x_i} \right), \quad i = 0, \ldots, n-1. \tag{66}$$

In the first type of problem $\eth F/\eth x_i$ and $\eth y_j/\eth x_i$ are known, while the unknowns are the $\eth F_1/\eth x_i \mathbf{y}^{e}$'s; in the second type of problem the unknowns are the $\eth y_j/\eth x_i$'s. From a computational point of view it seems that Boolean calculus may only be of some interest for solving the first type of problem: one has then to solve a system of linear equations and the classical methods relative to the real field may be used to obtain the solution. Let us give an example.

Example

$$F = x_1 x_2' x_3 x_4 + x_1' x_2 x_3 x_4 + x_1 x_2' x_3 x_4' + x_1 x_2 x_3 x_4 + x_1' x_2 x_3 x_4', \tag{67}$$

$$\begin{aligned} y_1 &= x_1 x_2' + x_1' x_2, \\ y_2 &= x_1 x_2, \end{aligned} \tag{68}$$

$$F = F_1(y_1 y_2 x_3 x_4). \tag{69}$$

Equation (32) becomes:

$$\begin{aligned} \frac{\partial F}{\partial x_1} &= x_3 \oplus x_2 x_3 x_4 = \frac{\partial F_1}{\partial y_1} \oplus x_2 \frac{\partial F_1}{\partial y_2} \oplus x_2 \frac{\partial F_1}{\partial(y_1 y_2)}, \\ \frac{\partial F}{\partial x_2} &= x_3 \oplus x_1 x_3 x_4 = \frac{\partial F_1}{\partial y_1} \oplus x_1 \frac{\partial F_1}{\partial y_2} \oplus x_1 \frac{\partial F_1}{\partial(y_1 y_2)}, \\ \frac{\partial F}{\partial x_3} &= (x_1 \oplus x_2) + x_1 x_2 x_4 = \frac{\partial F_1}{\partial x_3}, \\ \frac{\partial F}{\partial x_4} &= x_1 x_2 x_3 = \frac{\partial F_1}{\partial x_4}. \end{aligned} \tag{70}$$

The solutions of this linear system are

$$\begin{aligned} \frac{\partial F_1}{\partial y_1} &= x_3 y_1 + \lambda_1 y_1', \\ \frac{\partial F_1}{\partial y_2} &= y_1 (\lambda_3 \oplus x_3 x_4) + \lambda_2 y_1', \\ \frac{\partial F_1}{\partial(y_1 y_2)} &= \lambda_3, \\ \frac{\partial F_1}{\partial x_3} &= y_1 \oplus y_2 x_4, \\ \frac{\partial F_1}{\partial x_4} &= y_2 x_3, \end{aligned} \tag{71}$$

where λ_1, λ_2, λ_3 are arbitrary binary expressions. The equality conditions on the multiple derivatives allow us to obtain the following values for the arbitrary terms λ_i:

$$\begin{aligned} \lambda_1 &= x_3, \\ \lambda_2 &= x_3 x_4, \\ \lambda_3 &= 0. \end{aligned} \tag{72}$$

The function F_1 is finally obtained by means of its expansion (6), that is

$$F_1 = y_1 \frac{\partial F_1}{\partial y_1} \oplus y_2 \left(\frac{\partial F_1}{\partial y_2}\right)_{y_1=0} \oplus x_3 \left(\frac{\partial F_1}{\partial x_3}\right)_{y_1=y_2=0} \oplus x_4 \left(\frac{\partial F_1}{\partial x_4}\right)_{y_1=y_2=x_3=0}, \tag{73}$$

which becomes in view of (71) and (72):

$$F_1 = x_3\, y_1 \oplus x_3\, x_4\, y_2. \tag{74}$$

6. Conclusion

In this paper we have used the concepts of a total differential and of a total variation to derive various properties of Boolean functions. The attempt here was more to represent a set of results in a single consistent theory very near the classical differential calculus then develop techniques easy to apply in practice. However, some authors [2,3,10]) have pointed out that the use of Boolean calculus could lead to very simple methods for various kinds of problems arising in switching practice.

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