A CLASS OF MULTIVARIABLE
POSITIVE REAL FUNCTIONS
REALIZABLE BY THE BOTT-DUFFIN METHOD

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Abstract

Necessary and sufficient conditions are obtained for the realizability by the Bott-Duffin procedure of multivariable positive real functions which are of the first degree in all variables except one. This class of functions is wider than the ones considered by Saito and by Soliman and Bose, and also includes reactances.

The Bott-Duffin process for the transformerless synthesis of lumped one-ports is based on Richards' theorem which was extended to multivariable functions by Saito. For a class of multivariable positive real functions, Saito has also derived necessary and sufficient realizability conditions. His conditions assume a special form for the even part of the function and this automatically restricts his results to non-Foster functions. Recently, Soliman and Bose gave sufficient conditions for realizability by the Bott-Duffin procedure, which include Foster functions, but these conditions are not necessary.

In this paper, necessary and sufficient conditions are derived under which a multivariable function, of the first degree in all variables except one, can be realized by the Bott-Duffin process, i.e. by an iterative application of the multivariable Richards' theorem. This class is wider than that defined by Saito because in one of the variables the degree is allowed to exceed unity, and it is also wider than the class found by Soliman and Bose because the conditions imposed on the function are less stringent. Finally, necessary and sufficient conditions are also established for the Bott-Duffin synthesis of multivariable reactances of the first degree in all variables except one.

Let

\[ Z(\mu_1, \mu_2, \ldots, \mu_L) = n/d \]  

be a positive real multivariable function of degree \( k_j \) in the variable \( \mu_j \) (\( j = 1, 2, \ldots, L \)). Application of one cycle of the Bott-Duffin method with respect
to the variable $\mu_r$ leads to the realization of fig. 1 where $\alpha_r$ and $m_r$ are positive constants. By direct computation

$$Z_1 = m_r \frac{\alpha_r Z - m_r \mu_r}{\alpha_r m_r - \mu_r Z}, \quad (2)$$

which shows that the relation between $Z_1$ and $Z$ is precisely the one obtained when a Richards section in the variable $\mu_r$ is extracted from $Z$.

Saito’s multivariable extension of Richards’ theorem shows that $Z_1$ is positive real and of reduced degree $(k_r - 1)$ in the variable $\mu_r$ if and only if the prescribed multivariable function $Z$ satisfies

$$Z|_{\mu_r=\alpha_r} = -Z|_{\mu_r=-\alpha_r} = m_r, \quad (3)$$

independently of $\mu_j$ ($j = 1, \ldots, L; j \neq r$). Saito’s proof was actually given for the case $\alpha_r = 1$ but holds for any positive $\alpha_r$ by the transformation $\mu_r' = \mu_r/\alpha_r$. For $Z$ to be completely realizable by repeated application of the Bott–Duffin cycle it is thus necessary and sufficient that $Z$, and all subsequent functions $Z_1$, $Z_2$, $\ldots$, generated during the procedure, are positive real and verify a condition similar to (3) in one of the variables $\mu_j$ ($1 \leq j \leq L$). The following theorems, whose proofs are given in the appendix, will define a class of multivariable functions satisfying these requirements. Usually, there is only one value of the index $r$ for which condition (3) is fulfilled and this determines the order of extraction of the different variables.

In the following, the lower asterisk will denote a reversal of sign of all independent variables. Thus $n_* = n \{ -\mu_1, -\mu_2, \ldots, -\mu_L \}$. The notation $\delta_j(Z)$ is used to indicate the degree of $Z$ in the variable $\mu_j$.

**Theorem 1.** If (1) is positive real of the first degree in $\mu_1$ and such that

$$n_* d + n d_* = (\alpha_1^2 - \mu_1^2) \prod_{i=2}^{L} f_i(\mu_i^2), \quad (4)$$

where $f_i(\mu_i^2)$ is a single-variable even polynomial and where $\alpha_1$ is a positive constant, and if $Z|_{\mu_1=\pm_1}$ is a function of $\mu_2$, there exists a value $\mu_{20}$ such that $Z|_{\mu_2=\pm\mu_{20}}$ is independent of $\mu_1$. Moreover, $\mu_{20}$ is either a zero of $f_2$ or infinity and the latter case only occurs when $2\delta_2(d) > \delta_2(f_2)$ or $\delta_2(n) > \delta_2(d)$.

![Fig. 1. Bott–Duffin cycle.](image)
Theorem 2. If (1) is positive real with
\[ \delta_i(Z) = k; \quad \delta_i(Z) = 1 \quad (i = 2, \ldots, L) \] (5)
and satisfying
\[ n_* d + n d_* = f(\mu_1^2) \prod_{i=2}^{L} (\alpha_i^2 - \mu_i^2), \] (6)
where \( f(\mu_1^2) \) is an even polynomial in \( \mu_1 \) and \( \alpha_i > 0 \) \( (i = 2, \ldots, L) \), there exists a variable \( \mu_r \) \( (1 \leq r \leq L) \) such that \( Z|_{\mu_r = \pm \alpha_r} \) is independent of \( \mu_j \) \( (j = 1, \ldots, L; j \neq r) \) where, if \( r = 1 \), \( \alpha_1 \) designates a zero of \( f \) or infinity. In the latter case \( 2\delta_1(d) > \delta_1(f) \) or \( \delta_1(n) > \delta_1(d) \).

Theorem 3. If (1) is positive real satisfying (5), the synthesis of \( Z \) can be performed by the Bott–Duffin procedure if and only if
\[ Z + Z_* = K \prod_{s=1}^{k} (\alpha_{s}^2 - \mu_1^2) \prod_{i=2}^{L} (\alpha_i^2 - \mu_i^2)/d d_*, \] (7)
where \( \alpha_{s} \) \( (s = 1, \ldots, k) \) and \( \alpha_i \) \( (i = 2, \ldots, L) \) are positive constants and where \( K \neq 0 \).

Any single-variable reactance can be synthesized by the Bott–Duffin method\(^4\). For multivariable reactances the situation is however completely different since condition (3) is obviously not automatically fulfilled. On the other hand, a multivariable reactance satisfies \( Z + Z_* = 0 \), and at the outset it seems thus that the problem cannot be solved by a straightforward extension of the realizability conditions stated in theorem 3. This difficulty is circumvented by theorem 4.

Theorem 4. Necessary and sufficient conditions for the realizability by the Bott–Duffin method of a reactance (1) satisfying (5) is that there exist a variable \( \mu_i \) and a positive constant \( \alpha_i \) \( (2 \leq i \leq L) \) such that
(1) the function \( Z' = Z|_{\mu_i = \alpha_i} \), which is no longer a reactance, satisfies the realizability conditions of theorem 3,
(2) \[ \delta_j(Z') = \delta_j(Z) \quad (j = 2, \ldots, L; j \neq i). \] (8)

Corollary. Any two-variable reactance of the first degree in each variable is realizable by the Bott–Duffin procedure.

Indeed, since \( Z \) is a reactance of the first degree in \( \mu_1 \) it can be written as
\[ Z = (A + \mu_1 B)/(C + \mu_1 D), \]
where \( A, B, C \) and \( D \) are polynomials of the first degree in \( \mu_2 \) satisfying \(^5\).
\[ A = \varepsilon A_*, \quad B = -\varepsilon B_* \]
\[ D = \varepsilon D_*, \quad C = -\varepsilon C_* \]

\( \varepsilon = \pm 1 \) \hfill (9)

Since \( Z \) is also of the first degree in \( \mu_2 \) one has

\[ Z' = Z_{\mu_1=1} = (\alpha + \mu_2 \beta)/(\gamma + \mu_2 \zeta). \]

But then

\[ Z' + Z'_{\mu_1} = (\alpha \gamma - \mu_2^2 \beta \zeta)/(\gamma^2 - \mu_2^2 \zeta^2) \]

and \( Z' \) thus satisfies the realizability condition of theorem 3. Two cases must be considered: if in (9) \( \varepsilon = 1 \), then

\[ Z = (\alpha + \beta \mu_1 \mu_2)/(\gamma \mu_1 + \zeta \mu_2), \]

while for \( \varepsilon = -1 \)

\[ Z = (\alpha \mu_1 + \beta \mu_2)/(\gamma + \zeta \mu_1 \mu_2). \]

The corresponding realizations are given in fig. 2.

\[ \begin{align*}
\frac{\alpha}{\xi \mu_2} & \quad \frac{\beta}{\xi} \mu_1 \\
& \quad \frac{\beta}{\xi \mu_2} \\
\frac{B}{\gamma} \mu_2 & \quad \frac{\alpha}{\xi} \mu_1 \\
\end{align*} \]

\( \varepsilon = 1 \)

\[ \begin{align*}
\frac{\alpha}{\xi \mu_2} & \quad \frac{\beta}{\xi} \mu_1 \\
& \quad \frac{\beta}{\xi \mu_2} \\
\frac{B}{\gamma} \mu_2 & \quad \frac{\alpha}{\xi} \mu_1 \\
\end{align*} \]

\( \varepsilon = -1 \)

Fig. 2. Bott–Duffin synthesis of a two-variable reactance of the first degree.

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Appendix

A. Proof of theorem 1

If \( \delta_2(n) > \delta_2(d) \), then \( Z \) has a pole at infinity in \( \mu_2 \), independent of the other variables \(^6\) and the theorem is proved. In the following, only the case

\[ \delta_2(Z) = \delta_2(d) \]

will thus be considered.

Since \( Z \) is of the first degree in \( \mu_1 \), it can be expressed in the form

\[ Z = (A + \mu_1 B)/(C + \mu_1 D), \]

where \( A, B, C \) and \( D \) are polynomials in \( \mu_2, \ldots, \mu_L \). Condition (4) gives then

\[ BD_* + B_* D = \prod_{l=2}^{L} f_l(\mu_1^2), \]

(A.3)
It will now be shown that \( d \) cannot neither have a factor \((\alpha_1 - \mu_1)\) nor a factor \((\alpha_1 + \mu_1)\). Indeed, if \( d \) contained a factor \((\alpha_1 + \mu_1)\), then, because of (4), the same factor would be present in \( n d^* \). But \( d^* \) cannot have a factor \((\alpha_1 + \mu_1)\), else \( d \) would contain a factor \((\alpha_1 - \mu_1)\) which is impossible because \( Z \) is positive real. Similarly, \( n \) cannot have a factor \((\alpha_1 + \mu_1)\) because \( \delta_1(Z) = 1 \). Consequently, relation (A.4) implies
\[
[Z + Z^*_\pm]_{n_1 = \pm 1} = 0 \tag{A.5}
\]
or, by (A.2),
\[
\frac{A + \alpha_1 B}{C + \alpha_1 D} \equiv - \frac{A^* - \alpha_1 B^*}{C^* - \alpha_1 D^*} \quad \text{for all } \mu_2, \ldots, \mu_L, \tag{A.6}
\]
\[
\frac{A - \alpha_1 B}{C - \alpha_1 D} \equiv - \frac{A^* + \alpha_1 B^*}{C^* + \alpha_1 D^*} \quad \text{for all } \mu_2, \ldots, \mu_L. \tag{A.7}
\]
One considers the irreducible form of the rational function
\[
Z(\alpha_1, \mu_2, \ldots, \mu_L) + Z(\alpha_1, -\mu_2, \ldots, -\mu_L) = 2 n'/d', \tag{A.8}
\]
which in view of (A.2) can be written as
\[
2 n'/d' = \frac{A + \alpha_1 B}{C + \alpha_1 D} + \frac{A^* + \alpha_1 B^*}{C^* + \alpha_1 D^*}. \tag{A.9}
\]
If (A.9) is a function of \( \mu_2 \), one defines the points \((\mu_{02}, \ldots, \mu_{0L})\) and \((-\mu_{02}, \ldots, -\mu_{0L})\) as a pair of zeros of (A.9). Thus
\[
\frac{A_0 + \alpha_1 B_0}{C_0 + \alpha_1 D_0} = - \frac{A^*_{00} + \alpha_1 B^*_{00}}{C^*_{00} + \alpha_1 D^*_{00}}, \tag{A.10}
\]
where \( A_0 = A(\mu_{02}, \ldots, \mu_{0L}) \), \( A^*_{00} = A(-\mu_{02}, \ldots, -\mu_{0L}) \), etc. Comparison of (A.6) and (A.7) considered at \((\pm \mu_{02}, \ldots, \pm \mu_{0L})\) with (A.10) gives then, since \( \alpha_1 \neq 0 \),
\[
A_0/C_0 = B_0/D_0 \quad \text{and} \quad A^*_{00}/C^*_{00} = B^*_{00}/D^*_{00},
\]
and thus
\[
Z(\mu_1, \pm \mu_{02}, \ldots, \pm \mu_{0L}) \text{ is independent of } \mu_1. \tag{A.11}
\]
Consequently, by (A.5)
\[
Z(\mu_1, \mu_{02}, \ldots, \mu_{0L}) + Z(-\mu_1, -\mu_{02}, \ldots, -\mu_{0L}) \equiv Z(\alpha_1, \mu_{02}, \ldots, \mu_{0L}) + Z(-\alpha_1, -\mu_{02}, \ldots, -\mu_{0L}) = 0
\]
and taking (A.4) into account

\[ \prod_{i=2}^{L} f_i(\mu_i^2) \left[ d(\mu_1, \mu_2, \ldots, \mu_L) d(-\mu_1, -\mu_2, \ldots, -\mu_L) \right]^{-1} = 0. \]  

(A.12)

Since, by definition, all zeros of \( n' \) satisfy (A.12), all irreducible factors of \( n' \) are also factors of

\[ \prod_{i=2}^{L} f_i(\mu_i^2) \]

and hence can only depend on one variable. Thus

\[ n' = \prod_{i=2}^{L} n_i'(\mu_i^2), \]

where \( n_i' \) is a factor of \( f_i \). Consequently, since expression (A.9) is a function of \( \mu_2 \) it has a zero at \( \mu_2 = \mu_20 \), independently of the variables \( \mu_3, \ldots, \mu_L \), where \( \mu_20 \) is either a zero of \( n_2' \) or (and) infinity if \( \delta_2(d') > \delta_2(n') \). According to (A.11), \( Z|_{\mu_2 = \pm \mu_20} \) is then independent of \( \mu_1 \) and in the former case \( f_2(\mu_20^2) = 0 \) while in the latter case (A.12) implies \( 2\delta_2(d) > \delta_2(f_2) \).

If \( Z|_{\mu_1 = \alpha_1} \) is a function of \( \mu_2 \) but (A.9) is not, then

\[ Z|_{\mu_1 = \alpha_1} = \frac{n'(\mu_3, \ldots, \mu_L)}{d'(\mu_3, \ldots, \mu_L)} + \frac{n''(\mu_2, \ldots, \mu_L)}{d''(\mu_2, \ldots, \mu_L)}, \]

where \( n'/d' \) and \( n''/d'' \) are respectively the even and the odd parts of \( Z|_{\mu_1 = \alpha_1} \).

In view of (A.2),

\[ \frac{A + \alpha_1 B}{C + \alpha_1 D} = \frac{n' d'' + n'' d'}{d' d''}, \]

(A.13)

\[ \frac{A_* + \alpha_1 B_*}{C_* + \alpha_1 D_*} = \frac{n' d'' - n'' d'}{d' d''}. \]

(A.14)

Because of (A.1), \( \delta_2(d'') = \delta_2(Z|_{\mu_1 = \alpha_1}) \neq 0 \) and let then \( (\pm \mu_20, \ldots, \pm \mu_L0) \) denote a pair of zeros of the even (or odd) polynomial \( d'' \). By (A.13) and (A.14) one has, with the same notation as above,

\[ C_0 + \alpha_1 D_0 = 0 \quad \text{and} \quad C_{*0} + \alpha_1 D_{*0} = 0, \]

(A.15)

but (A.6) and (A.7) give then

\[ C_0 = D_0 = C_{*0} = D_{*0} = 0, \]

(A.16)
and thus \( Z(\mu_1, \pm \mu_{20}, \ldots, \pm \mu_{L0}) \) has a pole independent of \( \mu_1 \). Substitution of (A.16) in (A.3) yields

\[
\prod_{i=2}^{L} f_i(\mu_{10}^2) = 0
\]

and a reasoning similar to above shows then that

\[
d'' = \prod_{i=2}^{L} d''_i(\mu_i^2),
\]

where \( d''_i \) is a factor of \( f_i(\mu_i^2) \) and where \( d''_2 \) is not a constant. Consequently, if \( \mu_{20} \) is a zero of \( d''_2 \) then \( f_2(\mu_{20}) = 0 \) and \( Z|_{\mu_2=\mu_{20}} \) has a pole independent of \( \mu_1, \mu_3, \ldots, \mu_L \). Since \( Z \) is positive real, \( \mu_{20} \) lies necessarily on the imaginary axis of the \( \mu_2 \) plane.

B. Proof of theorem 2

The proof proceeds by repeated application of theorem 1. First it is observed that condition (6) implies \( \delta_i(n) = \delta_i(d) = 1 \) (\( i = 2, \ldots, L \)). Hence, if \( Z|_{\mu_2=\pm \alpha_2} \) is a function of \( \mu_3 \), then, by theorem 1, \( Z|_{\mu_3=\pm \alpha_3} \) is independent of \( \mu_2 \). If \( Z|_{\mu_3=\pm \alpha_3} \) is a function of \( \mu_4 \), then theorem 1 can again be applied to show that \( Z|_{\mu_4=\pm \alpha_4} \) is independent of \( \mu_3 \). Therefore, \( Z|_{\mu_4=\pm \alpha_4} = Z|_{\mu_4=\pm \alpha_4, \mu_3=\pm \alpha_3} \) and is thus also independent of \( \mu_2 \). Finally, there will exist a variable \( \mu_r \) (\( 2 \leq r \leq L \)) such that either \( Z|_{\mu_r=\pm \alpha_r} \) is a constant independent of all other variables \( \mu_j \) (\( j = 1, \ldots, L; j \neq r \)) or \( Z|_{\mu_r=\pm \alpha_r} \) is still a function of \( \mu_1 \) alone. In the latter case, theorem 1 indicates that there exists a value \( \mu_{10} \) such that \( Z|_{\mu_1=\pm \mu_{10}} \) is independent of \( \mu_r \) and thus independent of all variables \( \mu_i \) (\( 2 \leq i \leq L \)). The constant \( \mu_{10} \) is either a root of \( f \) or infinity, in which case \( 2\delta_1(d) > \delta_1(f) \) or \( \delta_1(n) > \delta_1(d) \).

C. Proof of theorem 3

Necessity

If \( Z_1(\mu_1) \) is a positive real function of the first degree in \( \mu_1 \), it is easy to show that (7) is satisfied.

Consider then the positive real function \( Z(\mu_1, \mu_r) \) of the first degree in \( \mu_1 \) and \( \mu_r \) if \( r \neq 1 \) and of the second degree in \( \mu_1 \) if \( r = 1 \). It is assumed that \( Z(\mu_1, \mu_r) \) is realizable by the Bott–Duffin method as shown in fig. 1 where \( \alpha_r \) and \( m_r \) are positive constants and where \( Z_1(\mu_1) \) satisfies (5) and (7) with \( k = 1 \).

By direct computation
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The latter expression is irreducible. Indeed, \( d \) cannot have a factor \((\alpha_r - \mu_r)\) because \( Z \) is positive real. On the other hand, a factor \((\alpha_r - \mu_r)\) cannot be present in \( d_* \), for else it would also be present in \( n_* \) and this would contradict the hypothesis on the degree of \( Z \). Consequently, \( d d_* \) cannot have a factor \((\alpha_r^2 - \mu_r^2)\).

The proof can now be completed by recurrence on the degree and the number of variables.

**Sufficiency**

It is seen that condition (7) implies \( \delta_j(n) = \delta_j(d) \) (\( j = 1, \ldots, L \)) and by theorem 2 there exists thus a variable \( \mu_r \) (\( 1 \leq r \leq L \)) such that

\[
Z|_{\mu_r = \alpha_r} = -Z|_{\mu_r = -\alpha_r} = m_r,
\]

where \( m_r \) is a positive constant and where \( \alpha_r = \alpha_{1,s} \) for some \( s (1 \leq s \leq k) \) if \( r = 1 \). A Bott–Duffin cycle can thus be applied as explained above and the remaining impedance \( Z_1 \) given in (2) is positive real and of reduced degree in \( \mu_r \). To complete the proof of sufficiency it remains to show that \( Z_1 \) satisfies a condition similar to (7). Indeed, by (2) and (7)

\[
Z_1 + Z_{1*} = \frac{K m_r^2 (\alpha_r^2 - \mu_r^2) \prod_{s=1}^{k} (\alpha_{1,s}^2 - \mu_1^2) \prod_{l=2}^{L} (\alpha_l^2 - \mu_l^2)}{(\alpha_r m_r d - \mu_r n) (\alpha_r m_r d_* + \mu_r n_*)},
\]

where numerator and denominator can be simplified by a factor \((\alpha_r^2 - \mu_r^2)^2\) as a consequence of (C.1).

**D. Proof of theorem 4**

**Necessity**

Since \( Z \) is realizable by Bott–Duffin, it is also realizable by an open- or short-circuited cascade of single-variable Richards sections. Consequently, there exists a Richards section in some variable \( \mu_i \) (\( 2 \leq i \leq L \)) such that all subsequent sections (if any) are in the variable \( \mu_i \) exclusively. If \( \alpha_i \) (\( \alpha_i > 0 \)) designates the transmission zero of that section, then \( Z' = Z|_{\mu_i = \alpha_i} \) is a resistively terminated cascade of Richards sections and hence it satisfies conditions (7) and (8).
Sufficiency

Since $Z$ is a multivariable reactance of the first degree in $\mu_i$ it can be written as

$$Z = (A + \mu_i B)/(C + \mu_i D), \quad \text{(D.1)}$$

where $A, B, C$ and $D$ are polynomials in $\mu_r$ ($r = 1, \ldots, L; r \neq i$) satisfying (9). By assumption, $Z' = (A + B)/(C + D)$ is realizable by the Bott–Duffin process and there exists thus a variable $\mu_r$ and a positive constant $\alpha_r (1 \leq r \leq L; r \neq i)$ such that

$$Z'_|_{\mu_r = \alpha_r} = -Z'_|_{\mu_r = -\alpha_r} = m_r. \quad \text{(D.2)}$$

Since $m_r$ is a positive constant, it follows by (9) that

$$[A/C]_{\mu_r = \pm \alpha_r} = [B/D]_{\mu_r = \pm \alpha_r},$$

which shows that $Z|_{\mu_r = \pm \alpha_r}$ is independent of $\mu_i$. Consequently, owing to (D.2),

$$Z|_{\mu_r = \alpha_r} = -Z|_{\mu_r = -\alpha_r} = m_r$$

and a Bott–Duffin cycle can thus be performed as shown in fig. 1. The remaining reactance $Z_1$ is given by (2), whence

$$Z_1|_{\mu_i = a_i} = m_r \frac{\alpha_r Z' - m_r \mu_r}{\alpha_r m_r - \mu_r Z'}.$$

This shows that $Z_1|_{\mu_i = a_i}$ is precisely the non-Foster impedance obtained by applying a Bott–Duffin cycle to $Z'$. Since $Z'$ is realizable by the Bott–Duffin method, the same must hold for $Z_1|_{\mu_i = a_i}$. The remaining reactance $Z_1$ satisfies thus the conditions of the theorem but with $\delta_i(Z_1) = \delta_i(Z) - 1$ while the degrees in all other variables remain unchanged.

Owing to (8), repeated application of the Bott–Duffin method will lead to a two-variable reactance $Z_n(\mu_1, \mu_i)$. One can assume that $\delta_i(Z_n) \neq 0$, for otherwise $Z_n$ is a single-variable reactance and thus automatically realizable by the Bott–Duffin method. If $Z_n|_{\mu_i = a_i}$ is a function of $\mu_1$, the procedure can be carried on until finally $Z_n|_{\mu_i = a_i}$ is a constant $m_i$, independent of $\mu_1$. But $Z_n + Z_n^* \equiv 0$ then implies $Z_n|_{\mu_i = -a_i} = -m_i$ and a Bott–Duffin cycle in the variable $\mu_i$ can thus be applied. Since $\delta_i(Z_n) = \delta_i(Z) = 1$, the remaining reactance $Z_{n+1}$ depends only on $\mu_1$ and is thus realizable by the Bott–Duffin method.
REFERENCES