PROPAGATION OF TRANSVERSE ELECTROACOUSTIC WAVES IN A PIEZOELECTRIC PLATE OF SYMMETRY $C_{6v}$ OR $C_{\infty v}$

by P. A. van DALEN

Abstract
Bleustein's theory of shear modes propagating in a piezoelectric plate is extended, and the resemblance between the two lowest plate modes and Bleustein-Gulyaev waves is described. The effective dielectric constant characterizing the electric properties at one surface of the piezoelectric plate is introduced. Experiments carried out on plates of piezoelectric ceramic produced results which are in good agreement with the theory presented. The experiments also showed two waves not predicted by the present theory, probably Lamb waves.

1. Introduction
Bleustein \(^1\) and Gulyaev \(^2\) have shown theoretically that in substances of symmetry $C_{6v}$ or $C_{\infty v}$, shear waves propagating in a half-space, along a stress-free surface and in a direction perpendicular to the symmetry axis lying in the surface, are electroacoustic surface waves. These waves have been called Bleustein-Gulyaev (BG) waves \(^*\).

In studying the behaviour of BG waves in detail it appears that in many situations the theory of BG waves has to be refined.

(1) The penetration depth of BG waves is determined by the wavelength, the electromechanical coupling factor, and the ratio $\varepsilon_1^{s}/\varepsilon_a$ between the relevant dielectric constant of the piezoelectric $\varepsilon_1^{s}$ and the dielectric constant of the adjacent medium. It is well known that in the case $\varepsilon_1^{s}/\varepsilon_a \gg 1$, e.g. when the surface is open, the penetration depth of the wave is larger than the wavelength by orders of magnitude. This implies that the velocity of the BG wave differs only slightly from the velocity of transverse bulk sound propagating perpendicularly to the symmetry axis. Hence, if it is desired to excite BG waves only and no bulk waves by means of an interdigital transducer, the transducer bandwidth has to be sufficiently narrow, which may be inconvenient.

(2) In ref. 4 it has been noted that the penetration depth of BG waves may even become negative when the piezoelectric is coupled electrically to a semiconductor subjected to a sufficiently strong transverse magnetic field. This would mean that the amplitude of the BG wave increases for increasing

\(^*\) Earlier a similar type of wave in crystals of symmetry $T_d$ has been described by Kaganov and Sklovskaya \(^3\).
distance to the surface. Hence, when this occurs one may no longer con-
sider the piezoelectric as a half-space.

A convenient way to take the effects owing to a strongly varying penetration
deepth into consideration is to study the propagation of transverse electro-
aoustic waves in a plate of the piezoelectric medium instead of in a half-space.

Bleustein \(^5\) has investigated the propagation of transverse electroacoustic
waves in piezoelectric plates of the aforementioned symmetry for two cases:
(a) both surfaces of the plate are shorted by a metal layer, (b) both surfaces
of the plate are bounded by vacuum. The dispersion relation of the plate
modes — with \(\omega\) and \(k\) being the angular frequency and the wavenumber of
the wave respectively, and \(d\) the thickness of the plate — is approximately
given by

\[
\frac{\omega^2}{v_s^2} = k^2 + \frac{n^2 \pi^2}{d^2}, \quad n = 0, 1, 2, \ldots,
\]

\(v_s\) being the piezoelectrically stiffened transverse sound velocity. Bleustein has
shown for the cases (a) and (b) how eq. (1) is modified by the piezoelectric
coupling. Furthermore he has noted for the case both surfaces are shorted
that the \(n = 0\) and the \(n = 1\) mode for \(d \to \infty\) acquire the velocity of BG
waves.

It is the purpose of the present paper to investigate the propagation of trans-
verse electroacoustic waves in a piezoelectric plate bounded on both sides by
media having a different dielectric constant; and to investigate the relationship
between the propagation of BG waves on the one hand, and the \(n = 0\) and
\(n = 1\) plate modes on the other hand.

The excitation efficiency of these plate modes (and of BG waves) is not con-
sidered here, but this is done in the next paper in this issue \(^6\).

In ref. 4 it has been shown that the electrical coupling between surface waves
in adjoining media can be described quite generally by introducing the concept
of the effective dielectric constant \(\varepsilon_{\text{eff}}\), characterizing the electrical properties
of a semi-infinite medium at the surface. The effective dielectric constant
of a piezoelectric plate, in which the transverse electroacoustic plate modes
propagate, is introduced in the present paper.

The electrical coupling between these plate modes and a semiconductor sub-
jected to a longitudinal drift field and a transverse magnetic field is studied
theoretically in another forthcoming paper \(^7\).

In order to demonstrate the resemblance between BG waves, and the \(n = 0\)
and \(n = 1\) plate modes, the theory of BG waves is resumed in sec. 2. In sec. 3
Bleustein's theory \(^5\) is extended by studying the modes in a piezoelectric plate
bounded on both surfaces by media having a different dielectric constant. In
sec. 4 the effective dielectric constant of the piezoelectric plate is calculated.
Section 5 describes measurements of the frequency dependence of the series resistance of interdigital transducers deposited on one side of a plate of piezoelectric ceramic (PXE5) showing that the frequency at which the plate modes propagate for a wavenumber dictated by the transducer period, is in excellent agreement with the theory. In the measurements two extra peaks occurred in the series resistance of the interdigital transducer, which could not be explained with the present theory. These peaks are presumably connected with the antisymmetric and the symmetric Lamb wave, excited by the electric field at the finger tips of the interdigital transducer.

2. Bleustein–Gulyaev waves in a semi-infinite medium

In the following a rectangular Cartesian \((x,y,z)\) coordinate system is chosen, with \(y = 0\) defining the surface of the piezoelectric substance of symmetry \(C_{6v}\) or \(C_{6v'}\). It is assumed that the material occupies the half-space \(y > 0\). The \(z\)-axis is parallel to the symmetry axis, and the \(x\)-axis is parallel to the propagation direction of the wave. The wave considered has only a mechanical displacement \(u\) in the \(z\)-direction. The wave is assumed to be of the form

\[
\exp\left[j (\omega t - k_x x - k_y y)\right],
\]

where \(\omega\) is the angular frequency, and \(k_x, k_y\) are the wavevector components in the \(x\)- and \(y\)-directions respectively. Hence the penetration depth is given by

\[
\delta = -\frac{1}{(\text{Im} \ k_y)}.
\]

For the present discussion the following equations of state of the piezoelectric medium are relevant:

\[
T_{xz} = c_44^E S_{xz} - e_{15} E_x,
\]

\[
D_x = e_{15} S_{xz} + \varepsilon_{11}^S E_x,
\]

\[
T_{yz} = c_44^E S_{yz} - e_{15} E_y,
\]

\[
D_y = e_{15} S_{yz} + \varepsilon_{11}^S E_y.
\]

The quantities \(c_44^E, e_{15}\) and \(\varepsilon_{11}^S\) are the appropriate elastic, piezoelectric and dielectric constants respectively. The equations (4)–(7) relate the stresses \(T_{xz}, T_{yz}\) and the components of the dielectric displacement \(D_x, D_y\) to the strains \(S_{xz}, S_{yz}\) and the electric fields \(E_x, E_y\). Since there is no \(z\)-dependence it follows from the definition of the strains that

\[
S_{xz} = \frac{\partial u}{\partial x}, \quad S_{yz} = \frac{\partial u}{\partial y}.
\]
Furthermore in the quasi-stationary approximation the electric field can be written as

\[ E_x = -\frac{\partial \phi}{\partial x}, \quad E_y = -\frac{\partial \phi}{\partial y}. \quad (9) \]

The stresses \( T_{xz} \) (4) and \( T_{yz} \) (6) have to satisfy the appropriate equation of motion

\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y}, \quad (10) \]

where \( \rho \) is the density of the material; \( D_x \) and \( D_y \) have to satisfy the Laplace equation

\[ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0. \quad (11) \]

Inserting (4), (6), (8), (9) into (10), and (5), (7), (8), (9) into (11), and using (2) yields the bulk dispersion relation

\[ (k_x^2 + k_y^2) \left( \frac{\omega^2}{v_s^2} - (k_x^2 + k_y^2) \right) = 0, \quad (12) \]

with

\[ v_s^2 = \frac{c_{44}^E (1 + e_{15}^2/c_{44}^E e_{11}^S)}{\rho} = \frac{c_{44}^D}{\rho}. \quad (13) \]

Here \( v_s \) is the piezoelectrically stiffened velocity of transverse sound, and \( c_{44}^D \) is the stiffened elastic constant. Equation (12) shows that in the piezoelectric medium the electrostatic mode — represented by the term between the first pair of brackets —, henceforth called the Laplace mode (denoted by superscript I), and the transverse sound mode — represented by the term between the second pair of brackets — (denoted by superscript II) are independent solutions of eqs (10) and (11).

A detailed analysis \(^4\) shows that

(1) The Laplace mode is purely electrostatic, i.e. it has no mechanical displacement:

\[ u^I = 0. \quad (14) \]

Furthermore it follows from (12) that

\[ k_y^I = -j k_x. \quad (15) \]
The mechanical displacement and the electrostatic potential of the transverse sound wave are related by

\[ u^{II} = \frac{\varepsilon_{11S}}{\varepsilon_{1S}} \varphi^{II}. \]  

(16)

Furthermore it follows from (5), (7), (8), (9) and (16) that the sound wave has no dielectric displacement. Equation (12) shows that

\[ k_y^{II} = -k_x \left( \frac{1}{v_s^2} \frac{\omega^2}{k_x^2} - 1 \right)^{1/2}. \]  

(17)

At the surface of the semi-infinite medium these modes have to obey the boundary conditions:

1. The surface is stress-free, the relevant boundary condition then reads

\[ T_{yz}^I (y = +0) + T_{yz}^{II} (y = +0) = 0. \]  

(18)

2. The dielectric displacement across the surface is continuous:

\[ D_y^I (y = +0) = D_y (y = -0) = \varepsilon_a E_y (y = -0), \]  

(19)

where \( \varepsilon_a \) is the dielectric constant of the adjacent medium.

3. The electric potential across the surface is continuous

\[ \varphi^I (y = +0) + \varphi^{II} (y = +0) = \varphi_a (y = -0). \]  

(20)

It can be shown after substituting (4)–(9) and (16) into (18)–(20), and using (2), and

\[ E_y (y = -0) = -k_x \varphi_a (y = -0), \]  

(21)

that

\[ k_y^{II} = -j \gamma^2 k_x \frac{\varepsilon_a}{\varepsilon_a + \varepsilon_{11S}}, \]  

(22)

where

\[ \gamma^2 = \frac{\varepsilon_{1S}^2}{\varepsilon_{44} \varepsilon_{11S}}. \]  

(23)

is the electromechanical coupling factor. Substitution of (22) into (17) yields the dispersion relation of the BG wave:

\[ \frac{\omega^2}{v_s^2} = k_x^2 \left[ 1 - \gamma^4 \left( \frac{\varepsilon_a}{\varepsilon_a + \varepsilon_{11S}} \right)^2 \right]. \]  

(24)
In ref. 4 it has been outlined that the wave phenomenon at the interface of two media due to the electrical coupling, can be generally described by introducing the effective dielectric constant of a medium at a surface. It is defined by

$$\varepsilon_{\text{eff}}(\omega, k_x) = \frac{D_p |k_{xr}|}{\varphi k_x k_{xr}},$$

(25)

with $k_{xr}$ being the real part of $k_x$. It has been shown also that the boundary conditions (19) and (20) at the interface between the two media can be written as

$$\varepsilon_{\text{eff}}^1(\omega, k_x) + \varepsilon_{\text{eff}}^2(\omega, k_x) = 0,$$

(26)

where the subscripts (1) and (2) refer to the different media on both sides of the interface. Equation (26) is the dispersion relation of the surface wave at the interface of the two media.

For the BG wave it is derived from the above equations \(^4\) that $\varepsilon_{\text{eff}}$ characterizing the piezoelectric medium is

$$\varepsilon_{\text{eff}}(\omega, k_x) = \varepsilon_{11}^S \left(1 \pm \frac{|k_{xr}|}{k_{xr}} \frac{\gamma^2}{(v^2/v_s^2 - 1)^{1/2}}\right)^{-1},$$

(27)

where $v = \omega/k_x$. In fig. 1 $\varepsilon_{\text{eff}}$ has been plotted as a function of $v/v_s$ for $\gamma^2 = 0.25$.

Fig. 1. $\varepsilon_{\text{eff}}$ as a function of the phase velocity for Bleustein-Gulyaev waves. Drawn curve: positive penetration depth; dashed curve: negative penetration depth.
In the figure the well-known quantities $v_0$ and $v_{oo'}$, as introduced by Ingebrigtsen\(^4,8\), are defined. The sign of the second term between big brackets of (27) suggests two possible values for $\varepsilon_{\text{eff}}$ at fixed $v/v_s$. A careful consideration\(^4\) shows that the negative sign corresponds to a positive penetration depth (denoted by a drawn curve in fig. 1) which is realistic physically, while the positive sign corresponds to a negative penetration depth (denoted by a dashed curve in fig. 1) which is not realistic physically. In sec. 4 it will be shown that the latter solution does not appear when one considers a piezoelectric plate.

3. Transverse electroacoustic waves in a piezoelectric plate of the Bleustein–Gulyaev symmetry

The equations in the present section refer to the same $(x,y,z)$ coordinate system as defined in the previous section. The surfaces of the plate are defined by $y = 0$ and $y = d$. The plate is bounded at the surface $y = 0$ by a medium having a dielectric constant $\varepsilon_a$ and at the surface $y = d$ by a medium having a dielectric constant $\varepsilon_b$.

In the case of a piezoelectric plate the Laplace mode and the transverse sound wave (12) are coupled at the surfaces $y = 0$ and $y = d$ by the boundary conditions. These are given for $y = 0$ by eqs (18)-(20), and for $y = d$ by

\[
T_{yz}^I (y = d) + T_{yz}^{II} (y = d) = 0, \tag{28}
\]

\[
D_y^I (y = d - 0) = D_y (y = d + 0) = \varepsilon_b E_y (y = d + 0), \tag{29}
\]

\[
\varphi^I (y = d - 0) + \varphi^{II} (y = d - 0) = \varphi_b (y = d + 0). \tag{30}
\]

In the present analysis two surfaces are involved, therefore the waves in the plate are of the form

\[
[A \exp (j k_y y) + B \exp (-j k_y y)] \exp [j (\omega t - k_x x)]. \tag{31}
\]

To be specific:

(I) The electric potential of the Laplace mode is of the form (see eq. (15))

\[
\varphi^I = [A \exp (-k_x y) + B \exp (k_x y)] \exp [j (\omega t - k_x x)], \tag{32}
\]

and

\[
u^I = 0. \tag{14}
\]

(II) The electric potential of the transverse sound wave is of the form

\[
\varphi^{II} = [C \exp (-j k_y^{II} y) + D \exp (j k_y^{II} y)] \exp [j (\omega t - k_x x)], \tag{33}
\]

where $k_y^{II}$ is given by (17). Furthermore

\[
u^{II} = \frac{\varepsilon_{I^5}}{e^5} \varphi^{II}. \tag{16}
\]
Insertion of (32), (14), (33) and (16) into the relevant piezoelectric equations of state (6) and (7), and using (8) and (9), shows

\[
T_{yz}^I = -k_x \varepsilon_{15} [A \exp (-k_x y) - B \exp (k_x y)] \exp [j (\omega t - k_x x)],
\]
\[
D_y^I = k_x \varepsilon_{11} s [A \exp (-k_x y) - B \exp (k_x y)] \exp [j (\omega t - k_x x)],
\]
\[
T_{yz}^{II} = -j k_y^{II} \varepsilon_{15} y^2 [C \exp (-j k_y^{II} y) - D \exp (j k_y^{II} y)] \exp [j (\omega t - k_x x)],
\]
\[
D_y^{II} = 0.
\]

Substitution of eqs (32), (33) and (34) into the boundary conditions (18), (19), (20) and (28), (29), (30); and using (21) and

\[
E_y (y = d + 0) = k_x \varphi_b (y = d + 0),
\]
gives six linear equations relating the quantities \(A, B, C, D, \varphi_a (y = -0)\) and \(\varphi_b (y = d + 0)\). Equating the secular determinant to zero yields the dispersion relation of the guided transverse electroacoustic waves

\[
(k_y^{II})^2 \left[ \varepsilon_a \varepsilon_b + \frac{1}{2} \varepsilon_{11} s (\varepsilon_a + \varepsilon_b) \tanh \left( \frac{1}{2} k_x d \right) + \frac{1}{2} \varepsilon_{11} s (\varepsilon_a + \varepsilon_b) \coth \left( \frac{1}{2} k_x d \right) + \gamma^2 k_x k_y^{II} \left[ \varepsilon_a \varepsilon_b \tan \left( \frac{1}{2} k_y^{II} d \right) \coth \left( \frac{1}{2} k_x d \right) + \frac{1}{2} \varepsilon_{11} s (\varepsilon_a + \varepsilon_b) \tan \left( \frac{1}{2} k_y^{II} d \right) \cot \left( \frac{1}{2} k_x d \right) - \varepsilon_a \varepsilon_b \tan \left( \frac{1}{2} k_y^{II} d \right) \cot \left( \frac{1}{2} k_x d \right) + \frac{1}{2} \varepsilon_{11} s (\varepsilon_a + \varepsilon_b) \cot \left( \frac{1}{2} k_y^{II} d \right) \right] - \gamma^4 k_x^2 \varepsilon_a \varepsilon_b = 0,
\]
where \(\omega\) is implicitly given by \(k_y^{II} (17)\). The solutions of (36), which must be determined numerically deviate slightly from (1) for ordinary values of \(\gamma^2\), and \(\varepsilon_a, \varepsilon_b\) positive *).

In the case \(\varepsilon_a = \varepsilon_b\), (36) reduces to

\[
\left( k_y^{II} - \gamma^2 k_x \frac{\varepsilon_a \tan \left( \frac{1}{2} k_y^{II} d \right)}{\varepsilon_a \tanh \left( \frac{1}{2} k_x d \right) + \varepsilon_{11} s} \right) \left( k_y^{II} + \gamma^2 k_x \frac{\varepsilon_a \cot \left( \frac{1}{2} k_y^{II} d \right)}{\varepsilon_a \coth \left( \frac{1}{2} k_x d \right) + \varepsilon_{11} s} \right) = 0.
\]

Essentially the same equation has been derived by Bleustein 5) for two particular cases (\(|\varepsilon_a| = \infty, \varepsilon_a = \varepsilon_0\)). The expression between the first pair of brackets represents plate modes with the displacement antisymmetric with respect to the centre plane of the plate \(y = \frac{1}{2} d\) (\(n\) in eq. (1) uneven), while the displacement is symmetric for the plate modes represented by the expression between the second pair of brackets (\(n\) even).

In the more general case \(\varepsilon_a \neq \varepsilon_b\) the modes are no longer symmetric or antisymmetric. Then the spatial pattern of the plate modes as given by (32) and (33) is described by the quantities \(A/B\) and \(C/D\). Expressions for these quantities are given in the appendix to this paper.

*) It will be shown in a future paper 7) that the solutions of (36) deviate strongly from (1) if the plate is bounded by a semiconductor subjected to a sufficiently strong transverse magnetic field.
In the following the solutions of (36) are considered in some detail.

(1) It can be shown analytically that for \( k_x d \to 0 \), (36) can be written as

\[
k_y^H \approx \frac{n \pi}{d}, \quad n = 0, 1, 2, \ldots
\]  

(38)

Substitution of this expression into (17) gives

\[
\frac{\omega}{v_s} \approx \frac{n \pi}{d}, \quad n = 0, 1, 2, \ldots
\]  

(39)

(2) For three particular cases the \( \omega - k \) relationship was determined numerically for a material with \( \gamma^2 = 0.25 \), \( \varepsilon_{11}^S = 100 \varepsilon_0 \). The three cases are:

(a) both surfaces bounded by vacuum \( (\varepsilon_a = \varepsilon_b = \varepsilon_0) \);

(b) one surface bounded by vacuum \( (\varepsilon_a = \varepsilon_0) \), one surface metallized \( (|\varepsilon_a| = \infty) \);

(c) both surfaces metallized \( (|\varepsilon_a| = |\varepsilon_b| = \infty) \).

The results, which are displayed in fig. 2 for the five lowest plate modes \( (n = 0, 1, 2, 3, 4) \), show three asymptotes for \( k_x d \to \infty \), denoted by \( p \), \( q \) and \( r \) respectively, which are given by

\[
\frac{\omega^2}{v_s^2} = k_x^2,
\]  

(40)

\[
\frac{\omega^2}{v_s^2} = k_x^2 \left( 1 - \frac{\gamma^4}{(101)^2} \right),
\]  

(41)

\[
\frac{\omega^2}{v_s^2} = k_x^2 (1 - \gamma^4),
\]  

(42)

the first two nearly coinciding. The numerical results show that in each case the plate modes \( n = 2, 3, \ldots \) approach the asymptote \( p \) as given by (40). The \( n = 0 \) and \( n = 1 \) modes behave differently in the three cases mentioned:

In case (a), the \( n = 0 \) and \( n = 1 \) modes approach the asymptote \( q \) as given by (41).

In case (b), the \( n = 0 \) mode approaches the asymptote \( r \) as given by (42), and the \( n = 1 \) mode approaches the asymptote \( q \) as given by (41).

In case (c), the \( n = 0 \) and \( n = 1 \) modes approach the asymptote \( r \) as given by (42).
Fig. 2. Dispersion relations of transverse electroacoustic plate modes in a piezoelectric plate of symmetry $C_{6v}$ or $C_{4v}$ ($y^2 = 0.25, \varepsilon_{11} = 100 \varepsilon_0$) for three cases:
(a) both surfaces bounded by vacuum;
(b) one surface bounded by vacuum, one surface metallized;
(c) both surfaces metallized.
The asymptotes $p$, $q$ and $r$ are given by eqs (40), (41) and (42) respectively.
This asymptotic behaviour of the plate modes can be understood analytically. One can show that from (36) for \(k_x d \to \infty\) the bulk equation for the propagation of transverse sound can be recovered. Furthermore two BG waves are found for \(k_x d \to \infty\): one at the surface \(y = 0\), and one at the surface \(y = d\). For \(k_x d \to \infty\), eq. (36) yields as solutions

\[
k_y^{(n)} = \frac{m \pi}{d}, \quad m = 1, 2, 3, \ldots, \tag{43a}
\]

\[
k_y^{(n)} = -j \gamma^2 k_x \frac{\varepsilon_a}{\varepsilon_a + \varepsilon_{11}s}, \tag{44a}
\]

\[
k_y^{(n)} = j \gamma^2 k_x \frac{\varepsilon_b}{\varepsilon_b + \varepsilon_{11}s}. \tag{45a}
\]

Substitution of (43a)–(45a) into (17) gives

\[
\frac{\omega^2}{v_s^2} = \frac{k_x^2}{d^2} + \frac{m^2 \pi^2}{d^2}, \quad m = 1, 2, 3, \ldots \tag{43b}
\]

\[
\frac{\omega^2}{v_s^2} = k_x^2 \left[ 1 - \gamma^4 \left( \frac{\varepsilon_a}{\varepsilon_a + \varepsilon_{11}s} \right)^2 \right], \tag{44b}
\]

\[
\frac{\omega^2}{v_s^2} = k_x^2 \left[ 1 - \gamma^4 \left( \frac{\varepsilon_b}{\varepsilon_b + \varepsilon_{11}s} \right)^2 \right]. \tag{45b}
\]

Equation (43b) corresponds for \(k_x d \to \infty\) to a continuous spectrum of transverse sound modes. Equations (44b) and (45b) correspond, as can be seen from (24), to a BG wave at the surface \(y = 0\) and a BG wave at the surface \(y = d\) respectively.

From the resulting equations (43)–(45) the numerical solutions described under (2) can be understood. Substitution of the appropriate values of \(\varepsilon_a\), \(\varepsilon_b\) and \(\varepsilon_{11}s\) reveals the asymptotes given by (40)–(42). The conclusion is that for \(k_x d \to \infty\) the \(n = 0\) plate mode (38) becomes the slowest BG wave of (44) and (45), and the \(n = 1\) plate mode becomes the other BG wave. Furthermore it follows from the numerical calculations that the modes numbered in (38) with \(n = 2, 3, \ldots\) become the modes numbered in (43) with \(m = 1, 2, \ldots\) for \(k_x d \to \infty\), approaching the asymptote given by (40).
4. The effective dielectric constant of the piezoelectric plate

In this section the effective dielectric constant of the piezoelectric plate at one surface of the plate is introduced. Its definition is given by eq. (25). The effective dielectric constant characterizes the electrical properties at one surface of the plate: it represents the piezoelectric properties of the plate, including the electrical coupling at the other surface between the adjacent medium and the piezoelectric plate.

There is no explicit expression for the effective dielectric constant of the plate, as there is for that of the half-space (27). However, the relationship between \( \varepsilon_{\text{eff}}(\omega, k_x) \) for the plate and the velocity of the transverse electroacoustic waves can be obtained numerically directly. For instance \( \varepsilon_{\text{eff}}(\omega, k_x) \) at the surface \( y = 0 \) can be determined in the following way: In

\[
\varepsilon_{\text{eff} 1}(\omega, k_x) + \varepsilon_{\text{eff} 2}(\omega, k_x) = 0, \tag{26}
\]

\( \varepsilon_{\text{eff} 1}(\omega, k_x) \) represents the effective dielectric constant of the plate, and \( \varepsilon_{\text{eff} 2}(\omega, k_x) = \varepsilon_a \). Hence,

\[
\varepsilon_{\text{eff} 1}(\omega, k_x) = -\varepsilon_a. \tag{46}
\]

Furthermore, insertion of the value of \( \varepsilon_a \) into the dispersion relation (36) yields \( v = \omega/k_x \). In other words, according to (46) there corresponds to each value of \( \varepsilon_a \) a value of \( \varepsilon_{\text{eff}} \) of the plate, and according to (36) there corresponds to each value of \( \varepsilon_a \) a value of \( v = \omega/k_x \) of the plate modes. Hence, there exists a unique relation between \( \varepsilon_{\text{eff}}(\omega, k_x) \) of the plate and the velocity of the guided transverse electroacoustic waves.

The relationship between \( \varepsilon_{\text{eff}} \) at the surface \( y = 0 \) and \( v = \omega/k_x \), thus obtained for a material with \( \gamma^2 = 0.25 \) and \( \varepsilon_{11} = 100 \varepsilon_0 \) at \( k_x d = \pi \), is shown in fig. 3 for two cases:

(a) \( \varepsilon_b = \varepsilon_0 \), surface \( y = d \) (almost) open,
(b) \( |\varepsilon_b| = \infty \), surface \( y = d \) shorted.

The figure shows \( \varepsilon_{\text{eff}} \) vs \( v/v_0 \) for the lowest plate modes. It can be seen from the figure and eq. (43) that for most values of \( \varepsilon_a \), \( \varepsilon_{\text{eff}} \) of the plate deviates only slightly from the vertical asymptotes. A characteristic feature is the switching of the modes from one asymptote to the next one for \( \varepsilon_{\text{eff}} = -\varepsilon_a \approx \varepsilon_{11} \); the physical meaning of this phenomenon will be discussed in a future paper.

The vertical asymptotes occur for \( |\varepsilon_{\text{eff}}| = |\varepsilon_a| = \infty \), i.e. for a shorted surface: they define for each mode \( v_0 \). The quantity \( v_{\infty} \) is the velocity belonging to an open surface, i.e. for \( \varepsilon_a = 0 \). Therefore for each mode \( v_{\infty} \) is found when \( \varepsilon_{\text{eff}} = 0 \).

The mode near the first asymptote for \( \varepsilon_{\text{eff}} < 0 \) (\( \varepsilon_a > 0 \)) is the \( n = 0 \) mode, the mode near the second asymptote is the \( n = 1 \) mode, etc. The positioning of the modes depends on the position of the asymptotes. From the argument
in the preceding two paragraphs it then follows that for increasing $k_x d$ the position of the asymptotes in figs 3a and 3b corresponds to the position of the modes as is shown in figs 2b and 2c respectively. In particular for $k_x d \to \infty$, it follows that

in case (a): $v_0$ of the $n = 0$ mode is given by (42), $v_0$ of the $n = 1$ mode is given by (41) and $v_0$ of the other modes is given by (40);

in case (b): $v_0$ of the $n = 0$ and the $n = 1$ mode is given by (42) and $v_0$ of the other modes is given by (40).

Finally, the relation between BG waves and the $n = 0$ mode can also be seen from comparing figs 1 and 3. The $\varepsilon_{\text{eff}}$ of a half-space and that of a plate for the $n = 0$ mode show resemblance. A remarkable distinction between the two is that the dashed curve in fig. 1, representing a wave with a negative penetration depth, does not appear in fig. 3. In fig. 3 it switches over to the next asymptote. It has been discussed in the previous section that the relation between BG waves and the $n = 1$ mode is only apparent for $k_x d \to \infty$, which is not shown in fig. 3.
5. Experimental results

In the present section the measurements of the frequency dependence of the series resistance of an interdigital transducer deposited on one surface of a plate of piezoelectric ceramic PXE5 are discussed. Henceforth this surface is called the upper surface, and the other surface is called the lower surface. The transducer fingers were parallel to the poling axis (z-axis), so that waves can be radiated in the x-direction. To prevent reflections at the edges of the sample obscuring the results, rhombic plates were taken such that reflected waves could not be detected by the transducer. The periodicity of the transducer was 480 μm, much larger than the grain size — of the order of 20 μm — of the ceramic material. The present paper deals with the propagation of modes in piezoelectric plates. Therefore, as regards the interpretation of the experiment the following remarks should be made.

(1) The periodicity of the transducer prescribes a fixed value of $k_x$. Furthermore, since the losses of the ceramic material are negligible in the range of applied frequencies (1–30 MHz), the bandwidth of the system is inversely proportional to the number of transducer finger pairs.

(2) At fixed $k_x$ the present theory predicts the frequency belonging to each plate mode.

(3) It may be expected that at the predicted frequency the plate mode absorbs energy from the transducer. The absorbed energy will correspond to a resonance peak in the series resistance of the transducer. The position of the peak is predicted by the present theory, and its width by the number of transducer finger pairs. The height of the peak is related to the amount of energy absorbed by the plate mode; it cannot be calculated with the present theory. In the next paper 6) a heuristic theoretical picture is presented to interpret the height of the peak.

Preliminary experiments 9) on plates of PXE5 of 500 μm thickness showed that the position of the resonance peaks, and the shift of the peak due to varying of $e_a$ and $e_b$, are in agreement with theoretical expectations. However, in the preliminary experiments the $n = 0$ and the $n = 1$ plate modes were not separated sufficiently.

Therefore, the resistance measurements were repeated for much thinner plates of PXE5, the thickness being 83 μm. The appropriate piezoelectric constants of the material were determined in the conventional way 10), the results being given in table I. The density of the material, measured at 300 K, was $7·57 \cdot 10^3$ kg/m$^3$.

The frequency dependence of the series resistance in the range 1–30 MHz was measured for two cases:

(a) both surfaces of the plate are bounded by vacuum;
(b) the upper surface is bounded by vacuum and the lower surface is metallized.
The results are shown in fig. 4, showing that according to expectations the resonance peaks are well separated.

Using the value of $k_x$ dictated by the transducer period, $d = 83 \, \mu m$ and the constants of PXE5 measured, one finds for the plate modes from (36) the frequencies indicated by drawn lines in the figure. From the figure it follows that a good agreement exists between theory and experiment as regards the positioning of the peaks. Special attention is drawn to the fact that metallizing a surface causes the peaks corresponding to the $n = 0$ and the $n = 1$ plate...
modes to shift. The figure shows that the magnitude of the measured shift agrees also with the shift predicted by the theory.

It was found that no resonance peaks are measured at the frequencies predicted by the theory for $3k_x$, and that resonance peaks did occur at the frequencies predicted for $5k_x$. The latter frequencies are indicated with a dashed line in the figure.

The $n = 0$ resonance peaks in fig. 4a show the well-known sidelobes, which could not be detected for the other resonance peaks. Superimposed on these sidelobes two extra resonance peaks are detected. These peaks indicated by a single and a double arrow correspond probably to the antisymmetric and symmetric Lamb waves respectively. It was also checked whether these peaks could be ascribed to non-linear effects, but changing the measuring voltage did not change the resonance peak.

As regards the excitation and propagation of these Lamb waves in piezoelectric plates of symmetry $C_{6v}$ or $C_{00v}$ in this particular geometry, the following remarks can be made:

1. Lamb waves propagating in the $x$-direction are coupled to the $z$-components of the electric field $E_z$ and the dielectric displacement $D_z$.

2. Since for a Lamb wave propagating freely in the $x$-direction $E_z = 0$ in the quasi-static approximation, its velocity is determined by the elastic constants $c^E$ of the material. Therefore, the theory for the free propagation of mechanical Lamb waves can be applied here. This theory is dealt with for isotropic substances for instance by Viktorov 11). The latter theory shows that for sufficiently thin plates there is only one symmetric and one antisymmetric Lamb wave.

3. For the Lamb wave $D_z$ is related to the relevant stresses $S_i$ by the piezoelectric equations of state, which are of the form

$$D_z = \sum_{i=1}^{3} e_{3i} S_i,$$

with $e_{31}$, $e_{32}$ and $e_{33}$ being the appropriate piezoelectric constants. Electric fields $E_z$ do exist in the interdigital transducer, in the regions between the ends of one set of fingers and the metal strip connecting the other set of fingers. In this array of regions having the periodicity of the transducer, Lamb waves can be generated. Because of the high dielectric constant of PXE5 the electric field is concentrated in the thin plate in the case $\varepsilon_a \ll \varepsilon_{11}^{S}$, $\varepsilon_b \ll \varepsilon_{11}^{S}$. Obviously in the case the lower surface is metallized and $E_z$ is shorted, the Lamb waves will not be launched.

To obtain a quantitative idea of the frequency of the Lamb waves, having a $k_x$ dictated by the transducer period, propagating in the $x$-direction in a plate of PXE5, one has to replace the appropriate elastic constants in the dispersion relation as given for instance in ref. 11. The numerical values of the constants
of PXE5 can be found approximately from those for the lead titanate–zirconate compositions with a Ti/Zr ratio of 45/55. It then follows that for 300 K the antisymmetric Lamb wave should be expected at about 2.8 MHz and the symmetric Lamb wave at about 7.3 MHz.

Acknowledgement

The author acknowledges the benefit of useful discussions with Dr C. A. A. J. Greebe and is indebted to Mr C. van den Bergh for preparing the samples and carrying out the experiments.

Eindhoven, February 1972

Appendix

For the sake of completeness the expressions for $A/B$ and $C/D$ are presented. From these, together with the solutions of the dispersion relation (36), the spatial and time dependence of $\varphi^I (32)$, $\varphi^{II} (33)$ and $u^{II} (16)$ can be found:

$$A = \frac{(1 - j \gamma^2 k_x/k_y^{II}) [-1 + \exp (k_x d + j k_y^{II} d)]}{B (1 - j \gamma^2 k_x/k_y^{II}) [1 - \exp (-k_x d + j k_y^{II} d)]} + \frac{\epsilon_{11} / \epsilon_a + (\epsilon_{11} / \epsilon_b) \exp (k_x d + j k_y^{II} d)}{\epsilon_{11} / \epsilon_a + (\epsilon_{11} / \epsilon_b) \exp (-k_x d + j k_y^{II} d)},$$

$$C = (1 - \epsilon_{11} / \epsilon_a + j \gamma^2 k_x/k_y^{II}) + (A/B) (1 + \epsilon_{11} / \epsilon_a - j \gamma^2 k_x/k_y^{II}),$$

$$D = (1 - \epsilon_{11} / \epsilon_a - j \gamma^2 k_x/k_y^{II}) + (A/B) (1 + \epsilon_{11} / \epsilon_a + j \gamma^2 k_x/k_y^{II}).$$

REFERENCES

2) Yu. V. Gulyaev, JETP Letters 9, 37, 1969.