HELICAL PIPE FLOW IN THE PRESENCE OF AN AXIAL PRESSURE GRADIENT

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Abstract

The problem discussed in this paper is closely related to the fluid-dynamic mechanism of regenerative pumps. The flow pattern within a pipe of circular cross-section is calculated by a perturbation method. The flow is induced by a row of blades, which is moving in the axial direction. Numerical results are presented, especially for large values of the pressure gradient.

1. Introduction

The study presented in this paper contributes to the explanation and calculation of the fluid-dynamic mechanism of the regenerative pump. A schematic drawing of such a pump is shown in fig. 1. The fluid passes through an annular channel and circulates repeatedly through the impeller blades. The channel between inlet and outlet is closed by a stripper. The main feature of the pump is that it develops in a single rotor high heads at low flow rates. Experiments indicate that a constant pressure gradient is created in the circumferential direction.

Several simple theories about this subject have been published, none of which is fully satisfactory. It is evident that a more detailed theoretical calculation of the flow pattern is needed. Fluid flow in a space of toroidal geometry, however, is very complicated and its calculation demands a large amount of
numerical effort 6). Therefore the present study will be restricted to the more simple geometry of a circular cylinder. If the radius of the torus is large, this assumption does not essentially affect the fluid-dynamic mechanism. Otherwise, it is always possible to correct afterwards for the influences of centrifugal and Coriolis forces.

The flow in the cylinder is calculated by imposing the ideal condition of shock-free entry and exit of the impeller blades. The number of blades is assumed to be infinite. Together with the assumption of a constant pressure gradient, this leads to a fluid-velocity field that is independent of the axial coordinate. In this way the original problem can be reduced to a two-dimensional mathematical problem in the interior of a circle.

Further simplifying assumptions are justified if one specific kind of forces is dominating. If the viscous forces dominate the inertia forces, the leading term of the solution can be described by a potential and a bi-potential equation. Another possibility is that the viscous forces can be neglected. This leads to a potential equation which determines the velocities in the cross-section plane of the cylinder. The axial velocity follows from a first-order differential equation. These equations can be solved by using complex variables and conformal transformations. The model chosen in this paper takes into account both viscous and inertia forces. It is, however, restricted to flow patterns without a radial velocity component. For this reason a special class only of impeller blades will be considered. The present problem bears some resemblance to the magneto-hydrodynamic pipe flow of an electrically conducting viscous fluid in the presence of a magnetic field 7).

2. Formulation of the problem

Figure 2a shows the system of coordinates for the consideration of the fluid motion through a pipe of circular cross-section. The coordinate in the axial direction of the pipe is denoted by \( \bar{z} \). The velocity components corresponding to the coordinates \( (\bar{r}, \theta, \bar{z}) \) are \( (U, V, W) \). The infinite number of blades is assumed to occupy a sector of the circular cross-section (fig. 2b). A special case is represented in fig. 2c, where the sector angle equals zero. Here the sector with the blades is degenerated into a line of discontinuity in the velocity and pressure fields of the fluid flow.

Fig. 2. The coordinate system.
The present analysis will be restricted to flow patterns that have as a common feature a zero radial velocity \((U = 0)\) and a constant axial pressure gradient. This leads to a velocity field that is independent of the direction \(\tilde{z}\).

The following dimensionless quantities are introduced:

\[
\begin{align*}
  r &= \tilde{r}/a, \quad z = \tilde{z}/a, \quad v = V/S, \quad w = W/S, \\
p &= \frac{P}{\rho S^2}, \quad \mu = \frac{\nu}{aS}, \\
C_0 &= \frac{dp}{dz}, \quad B = \frac{1}{2\mu} \frac{dp}{d\theta}, \\
K &= \frac{K_z}{\rho a S^2}, \quad Q = \frac{Q_0}{\pi a^2 S},
\end{align*}
\]

where \(S\) is the speed of the impeller blades, \(P\) the pressure and \(\nu\) the viscosity of the fluid; \(K_z\) is the force per unit of length in the direction \(z\) applied to the fluid by the row of blades, and \(Q_0\) is the volume rate of flow.

The governing differential equations for \(v\) and \(w\) are then

\[
\begin{align*}
  \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} - \frac{2B}{r} &= 0, \quad (2.1) \\
  \mu \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) - \frac{v}{r} \frac{\partial w}{\partial \theta} &= C_0. \quad (2.2)
\end{align*}
\]

The solution for the tangential velocity \(v\), which satisfies the boundary condition \(v(1) = 0\), is easily obtained:

\[
v = E \left( \frac{1}{r} - r \right) + B r \ln r, \quad (2.3)
\]

where \(E\) is an as yet unknown constant of integration.

Apart from the condition \(w(1) = 0\), boundary conditions are imposed on \(w\) at the entry and the exit of the blades that have to guarantee a smooth passage of the flow. The leading and trailing edges of the blades are supposed to be on radial lines. Then the boundary conditions follow from the velocity diagram shown in fig. 3:

\[
\begin{align*}
  s_1 &= \tan \beta_1 = \frac{v}{w_1 - 1}, \quad (2.4) \\
  s_2 &= \tan \beta_2 = \frac{v}{w_2 - 1},
\end{align*}
\]

where the suffices 1 and 2 correspond to the trailing edge and leading edge.
Fig. 3. The velocity diagram at the impeller blade.

respectively. The flow through the blades is linked to the channel flow by a condition for the pressure difference across the blades:

\[ p_2 - p_1 = 2 (2\pi - \alpha) \mu B \]
\[ = \frac{1}{2} [(w_1 - 1)^2 - (w_2 - 1)^2] - k [(w_2 - 1)^2 + v^2], \]  

where the last term represents the viscous loss of the flow after passage of the blades. The value of the constant \( k \) depends on the geometry of the blades and can be found in ref. 8.

An exact solution of (2.2) can be obtained for the special case where \( B = 0 \). It is possible to construct a Green's function in terms of a Fourier-Bessel series. The function \( w \) can be expressed explicitly by application of Green's theorem. The series of the solution, however, converge very poorly. Therefore, another approach is chosen here, which is moreover not restricted to the case \( B = 0 \).

The approach consists of a perturbation method with respect to the small parameter \( \mu \). The solution obtained in this way consists of a core solution or outer expansion and a boundary-layer solution or inner expansion, which are matched together to a uniformly valid solution.

First, however, the inviscid part of the core solution will be derived. This rather simple solution is used to discuss some general physical aspects of the flow.

3. The inviscid core solution

The only inviscid velocity field that can exist in the absence of the radial component is determined by

\[ v = \frac{E}{r}, \]
\[ \frac{v \partial w}{r \partial \theta} = -C_0. \]  

(3.1)
The solution satisfying the first boundary condition of (2.4) results in

$$w = 1 + \frac{E}{s_1 r} \frac{C_0 r^2}{E} \theta. \tag{3.2}$$

The second boundary condition can be satisfied only if $s_2$ belongs to the one-parameter family of curves characterized by

$$s_2 = \frac{s_1 E^2}{s_1 r E + E^2 - s_1 r^3 C_0 (2\pi - \alpha)}. \tag{3.3}$$

The choice of a particular value for $s_2$ determines the unknown strength $E$ of the potential vortex. The action of this vortex $E$ is analogous to that of the circulation around an airplane wing and guarantees a shock-free flow at the impeller blades.

The second equation of (3.1) is the mathematical formulation of the momentum theorem. The momentum per unit mass flow changes and is converted into the action of the pressure gradient. Integration of (3.1) leads to

$$C_0 = \frac{v H}{(2\pi - \alpha) r}, \tag{3.4}$$

where

$$H = w(r, 2\pi - \alpha) - w(r, 0).$$

If the dimensionless pressure difference across a pipe of length $L$ is denoted by $\Delta p$, (3.4) can be written as

$$\Delta p = \frac{t H 2\pi}{2\pi - \alpha}, \tag{3.5}$$

where

$$t = \frac{v L}{2 \pi \bar{r}}.$$  

The force per unit area exerted by the impeller on the medium in the direction $z$ equals the momentum transferred to the medium per unit time:

$$f = \varrho S^2 v H = (2\pi - \alpha) \bar{r} \frac{\Delta P}{L}.$$

Integration in the radial and the axial direction results in

$$F = \Delta P a^2 (\pi - \alpha/2),$$

where $F$ is the total force applied by the impeller.

From (3.5) it can be concluded that the magnitude of $\Delta p$ is determined by two factors. In order to achieve a high pressure difference it is necessary that
the product of these factors is large. The first factor \( t \) represents the number of times that a fluid particle passes through the impeller blades. The second factor \( H \) is a measure of the reversal of the momentum flow within the blades. An increase in the second term causes an increase of the resistance for the flow through the blades. This effect restricts the magnitude of the first term \( t \). In this way it becomes clear that a maximum value for \( \Delta p \) is obtainable with an optimal design of the blade geometry.

4. The uniformly valid approximate solution

In this section an approximate solution of the problem will be derived for a special class of \( s \)-functions. This class is defined as

\[
s_1 = \frac{t}{1 + \sigma},
\]

\[
s_2 = \frac{\tau}{\sigma},
\]

where

\[
\tau = \frac{v}{H},
\]

\[
\sigma = \frac{H^2 - [H^4 + 2k H^2 (H^2 - 4H - 4k^2 v^2 - 8 \pi \mu B)]^{1/2}}{2k H^2},
\]

\[
v = E \left( \frac{1}{r} - r \right) + B r \ln r,
\]

\[
H = W_2 - W_1.
\]

The formulation above is chosen in order to satisfy the conditions (2.4) and (2.5). Later on, when the approximate solution of \( w \) will be known, it will appear that \( H \) can be expressed as a function of \( E \) and \( B \). Therefore, the functions (4.1) depend on the choice of the two parameters \( E \) and \( B \).

After the introduction of the two constants,

\[
\varepsilon = \frac{\mu}{2E - B} \quad \text{and} \quad C = \frac{C_0}{2E - B},
\]

the boundary-value problem for \( w \) can be formulated as follows:

\[
\varepsilon \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) - \frac{v}{r (2E - B)} \frac{\partial w}{\partial \theta} = C,
\]

with

\[
w(1, \theta) = 0,
\]

\[
w(r, 0) = w_1 = 1 - s_2 H/(s_2 - s_1),
\]

\[
w(r, 2\pi - \alpha) = w_2 = w_1 + H.
\]
It appears to be convenient to introduce a new coordinate system \((x, \theta)\) based on the relation
\[
r = e^{-x}.
\]
(4.7)

Equation (4.5) is then replaced by
\[
\varepsilon \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \theta^2} \right) - \frac{v e^{-x}}{2E - B} \frac{\partial w}{\partial \theta} = C e^{-2x}.
\]
(4.8)

The derivation of the approximate solution includes the determination of an outer and an inner solution and a matching of these to a uniformly valid approximation. Only the lowest-order terms in the small parameter \(\varepsilon\) will be maintained.

The outer solution originates from
\[
-\frac{v}{2E - B} \frac{\partial w}{\partial \theta} = C e^{-x},
\]
and becomes
\[
w_0 = w(x, 0) - \frac{(2E - B) C e^{-x}}{v}.
\]
(4.9)

The inner solution or boundary-layer solution is valid near \(x = 0\) and \(\theta = 2\pi - \alpha\). Here a new assumption enters the theory concerning the boundary layer at \(\theta = 2\pi - \alpha\). The velocity \(w(x, 2\pi - \alpha)\) is chosen in such a way that the outer solution matches the required velocity at \(\theta = 2\pi - \alpha\). Because of this assumption the value of \(H\) in (4.6) is determined. What remains to be calculated is the inner solution near \(x = 0\).

A stretching of the coordinate \(x\) is carried out:
\[
x = \varepsilon^{1/3} \xi,
\]
(4.10)

and \(w\) and \(v\) are expanded in a power series as follows:
\[
w_t = g_0 e^{-1/3} + g_1 + e^{1/3} g_2 + \ldots,
\]
\[
v = e^{1/3} \xi \frac{dw}{dx}(0) + \frac{1}{2} e^{2/3} \xi^2 \frac{d^2 w}{dx^2}(0) + \ldots.
\]
(4.11)

By substituting (4.10) and (4.11) into (4.8) and making the coefficients of order unity and order \(\varepsilon^{1/3}\) vanish, the following differential equations are obtained:
\[
\frac{\partial^2 g_0}{\partial \xi^2} - \xi \frac{\partial g_0}{\partial \theta} = C,
\]
(4.12)

\[
\frac{\partial^2 g_1}{\partial \xi^2} - \xi \frac{\partial g_1}{\partial \theta} = -2 C \xi - \xi^2 (1 - A) \frac{\partial g_0}{\partial \theta}, \quad g_1(0, \theta) = g_1(\xi, 0) = 0,
\]
(4.13)
where

\[ 2A = \frac{d^2 v}{dx^2}(0) \left/ \frac{dv}{dx}(0) = \frac{2B}{2E - B} \right. \]

Again new coordinates are introduced:

\[ u = \xi^{3/2} \quad \text{and} \quad g = u^{1/3} h. \]

The operator \( N \) denotes

\[ N = \frac{\partial^2}{\partial u^2} + \frac{1}{u} \frac{\partial}{\partial u} - \frac{1}{9u^2} - \frac{\partial}{\partial \theta}. \]

The system of eqs (4.12) and (4.13) can be written as

\[ N(h_0) = \frac{C}{u}, \quad \text{and} \quad h_0(0, \theta) = h_0(u, 0) = 0, \]

\[ N(h_1) = -\frac{8}{3} C u^{-1/3} - \frac{8}{3} (1 - A) u^{2/3} \frac{\partial h_0}{\partial \theta}, \quad h_1(0, \theta) = h_1(u, 0) = 0. \]

These equations can be solved by applying a Hankel K-transform with respect to \( u \) on both sides. After careful elaboration the result becomes

\[ h_0 = -\frac{4}{3} C \int_0^\infty [1 - \exp (-\frac{8}{3} \theta t^2)] J_{1/3}(u t) t^{-2} dt, \]

\[ h_1 = \frac{C}{15} \int_0^\infty [1 - \exp (-\frac{8}{3} \theta t^2)] t^{-2} \left( \frac{2^{8/3} (7 + 3A) t^{-2/3}}{3 I(\frac{4}{3})} J_{1/3}(u t) + \right. \]

\[ \left. -2 (1 - A) u^{5/3} t J_{4/3}(u t) \right) dt, \]

\[ w_1 = e^{-1/3} u^{1/3} h_0 + u^{1/3} h_1. \]

Another expression for the inner expansion is

\[ w_1 = C \int_0^\beta \left( -\frac{4}{3} e^{-1/3} u^{4/3} I(\frac{4}{3}, 0, \alpha) + \frac{2^{8/3} (7 + 3A) u^2}{45 I(\frac{4}{3})} I(\frac{4}{3}, -\frac{2}{3}, \alpha) \right) d\alpha + \]

\[ -\frac{3}{16} (1 - A) C u^2 [\frac{4}{3} - I(\frac{4}{3}, -1, \beta)], \]

with

\[ \beta = 9\theta/4u^2 \]
and

\[ I(v, \mu, \alpha) = \int_0^\infty \exp(-\alpha s^2 + \mu \ln s) J_v(s) \, ds = \]

\[ \Gamma\left(\frac{v + \mu + 1}{2}\right) \exp\left(-\frac{v + \mu + 1}{2} \ln \alpha\right) \times \]

\[ \times \, _1F_1\left(\frac{\mu + v + 1}{2}; v + 1; -\frac{1}{4\alpha}\right)/\Gamma(v + 1), \tag{4.15} \]

where \( _1F_1 \) is a confluent hypergeometric function. By applying the asymptotic expansion of this function \(^9\) it can be proved that \( w_i \) behaves as

\[ w_i = -C \theta/x + C (1 + A) \theta, \quad \text{for} \quad x \to \infty. \]

Near \( x = 0 \) the outer expansion has the behaviour

\[ w_0 = -C \theta/x + C (1 + A) \theta, \quad \text{for} \quad x \to 0. \]

Hence it can be concluded that the matching condition is inherently satisfied.

The uniformly valid solution can be constructed from the addition of \( w_i \) and \( w_0 \) minus their common part:

\[ w = -C (2E - B) e^{-x} \theta/v + w(x, 0) + C \theta/x - C (1 + A) \theta + \]

\[ + \frac{2^8/3 (7 + 3A) t^{-2/3}}{45 \Gamma(\frac{4}{3})} J_{1/3}(ut) - \frac{8}{13} (1 - A) u^{1/3} J_{1/3}(ut) \right) dt. \tag{4.16} \]

5. Calculation of the applied force and the rate of flow

In order to facilitate the calculation of the velocity field the expression (4.16) can be reformulated by using (4.15) and the relation

\[ _1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(b - a) \Gamma(a)} \int_0^1 t^{a-1} (1 - t)^{b-a-1} \, dt. \]

After substitution of (4.1) and (4.6) into (4.16) the result becomes

\[ w(x, \theta) = 1 - C_0 \frac{2\pi - \alpha}{2E - B} \left\{ [m_1(x) + m_2(x, h)] + \right. \]

\[ -\phi^3 \left[ m_2(x, h) + m_2\left(x, \frac{h}{\phi}\right) \right], \]

\[ (\sigma + 1) [m_1(x) + m_2(x, h)] + \]

\[ -\phi^3 \left[ m_2(x, h) + m_2\left(x, \frac{h}{\phi}\right) \right]. \]
where the following notations are introduced:

\[ \phi^3 = \frac{\theta}{2\pi - \alpha}, \]

\[ h = [9 (2\pi - \alpha) \epsilon]^{-1/3}, \]

\[ m_1(x) = \frac{(2E - B) e^{-x}}{2E \sinh(x) - B \times e^{-x}} - \frac{1}{x} + 1 + A, \]

\[ m_2(x) = -T_1(h \times) - 9^{1/3} h T_2(h \times), \]

\[ T_1(y) = (1 - A) \left( -\frac{9}{10} y^3 + \frac{81 \Gamma(\frac{1}{3})}{40 \pi} a_1 y^5 \right) + \]

\[ + \frac{3}{10 \Gamma(\frac{1}{3})} (7 + 3\Lambda) (2a_1 + a_3) y, \]

\[ T_2(y) = \frac{9 \Gamma(\frac{1}{3})}{8\pi} (2a_4 - a_5) y, \]

\[ a_1 = \int_0^1 (1 - t^{3/2})^{2/3} \exp(-y^3 t^{3/2}) \, dt, \]

\[ a_2 = \int_0^1 \exp(-y^3 t^3) \, dt, \]

\[ a_3 = \int_0^1 \exp(-y^3 t^{-3/2}) \, dt, \]

\[ a_4 = \int_0^1 \exp(-y^3 t^{-3}) \, dt, \]

\[ a_5 = \int_0^1 \exp(-y^3 t^{3/2}) [(1 - t^{3/2})^{2/3} - 1] t^{-3/2} \, dt. \]

The non-dimensional force per unit of length in the direction \( z \) applied to the fluid by the row of blades is given by

\[ K = \int_0^1 v (w_2 - w_1) \, dr. \]

Using the functions defined before, this leads to

\[ K = -C_0 \frac{2\pi - \alpha}{2E - B_0} \int \frac{m_1(x) + m_2(x, h)}{v e^{-x}} \, dx. \]

The expression for the non-dimensional rate of flow is

\[ Q = \frac{1}{\pi} \int_0^1 \frac{1}{\phi^3} \int_0^1 w r \, dr. \]
After some calculations this can be reduced to

\[
Q = 1 - C_0 \frac{2\pi - \alpha}{2E - B} (q_1 + q_2 + q_3 + q_4),
\]

\[
q_1 = -\int_0^\infty (1 + 2\sigma) m_1(x) e^{-2x} \, dx,
\]

\[
q_2 = -2 \int_0^\infty (1 + \sigma) m_2(x, h) e^{-2x} \, dx,
\]

\[
q_3 = -2160 \int_0^\infty m(x) \left[ T_1\left(\frac{hx}{2}\right) + 3^{-1/3} \frac{hx}{2} T_2\left(\frac{hx}{2}\right) \right] \, dx,
\]

\[
q_4 = -\int_0^\infty \exp\left(-2x/h\right) \left( 3^{2/3} T_2(x) - \frac{\exp(-1/x)}{x} \right) \, dx - 2 K_0(2(2/h)^{1/2}),
\]

\[
m(x) = \frac{1}{720} \int_0^1 \exp(-x t) t^6 \, dt,
\]

where \(K_0\) is a modified Bessel function.

It would be natural to calculate \(B, s_2\) and \(\sigma\) from given values of \(k, s_1\) and \(E\). It is, however, much easier to obtain \(k, s_1\) and \(s_2\) as functions of the variables \(B, E\) and \(\sigma\). In order to get an insight into the effect of the various variables the functions \((Q - 1)/C_0, K/C_0, k, s_1, s_2\) and \(\omega\) are calculated for different values of \(E, B, \sigma\) and \(\mu\). This is achieved by a computer program which is based on the formulas derived before and on the expressions

\[
k = \frac{4\pi \mu B - H^2 (\sigma + \frac{1}{2})}{\sigma^2 H^2 + \nu^2},
\]

\[
s_1 = \frac{\nu}{w_1 - 1}, \quad s_2 = \frac{1 + \sigma}{\sigma - s_1}.
\]

The numerical calculations represented in this paper have been carried out for a sector angle \(\alpha = 0\). With the exception of the results in fig. 4, \(\mu\) was taken to be \(\mu = 10^{-2}\). For the sake of convenience the function \(\sigma\) has been assumed to be independent of \(r\).

The effect of \(\mu\) on \(K\) and \(Q\) is shown in fig. 4. It must be realized that the increase of \(K\) at a smaller value of \(\mu\) (fig. 4a) is connected with a simultaneous increase of \(Q\) (fig. 4b).

The influence of \(\sigma\) for different values of the parameters \(E\) and \(B\) is represented in fig. 5. Here a larger value of \(\sigma\) corresponds to a larger resistance coefficient \(k\) and to a smaller value of \(C_0\) at a particular \(Q\).

For a particular case \(\sigma = -0.5, E = 0.3, B = 0.15\) and \(Q = 0.175\) the
Fig. 4a. Effect of the parameter $\mu$ on $K/C_0$.

Fig. 4b. Effect of the parameter $\mu$ on $(Q - 1)/C_0$. 
functions $k$, $s_1$ and $w$ are plotted in fig. 6. The resulting values for $C_0$ and $K$ are $C_0 = 1.1$ and $K = 1.7$. The order of magnitude of the calculated value for $k$ is in agreement with published values elsewhere 8). The graph 6c gives an impression of the flow-back that results at this high pressure.

Fig. 4c. Effect of the parameter $\mu$ on $K/(Q - 1)$.

Fig. 5a. The function $K/C_0$. 

\[ K \]
Fig. 5b. The function \((Q - 1)/C_0\) for \(\sigma = -0.5\).
Fig. 5c. The function \((Q - 1)/C_0\) for \(\sigma = -0.52\).
Fig. 5d. The function \((Q - 1)/C_0\) for \(\sigma = -0.55\).

Fig. 6a. The resistance coefficient \(k\).
Fig. 6b. The blade angle $\beta_1$.

Fig. 6c. The axial velocity $w$. 
6. Conclusion

Strictly speaking, the present theory is valid for laminar flow only. It can be applied, however, to the more practical case of turbulent flow as well, if the velocities are interpreted as mean values and the turbulent stresses as apparent viscous stresses. It is possible to define a so-called eddy viscosity which depends on the coordinates and is to be determined by experiment. If the approximation of a constant eddy viscosity is accepted, the theoretical results of sec. 5 remain valid for a turbulent flow. Using experimental data it can be concluded that a value for the eddy viscosity of \( \mu = 10^{-2} \) corresponds to a real viscosity value of \( \mu = 10^{-4} \). Therefore, the theoretical results calculated for this value can be applied to the case of a turbulent air-flow in a pipe with a diameter of 1 cm and a speed of the impeller of about 15 m/s.

The value \( \sigma = -0.5 \) corresponds to an impeller that is characterized by identical diagrams for the relative velocity vectors at the leading and trailing edges of the blades. The pressure difference across the blades is in equilibrium with the pressure loss due to viscous friction.

The tangential velocity field has a singular behaviour at \( r = 0 \). In a real flow this phenomenon cannot exist. Therefore, the theory is not valid within a small region around \( r = 0 \).

For situations where \( 2E - B \) becomes zero the theory breaks down. Then another procedure in the series expansion must be followed, because the derivative of the velocity \( v \) becomes zero.

The calculation in sec. 5 refers to a single point in the pump characteristic. It is difficult to calculate the other points for fixed values of the impeller angles, because no smooth flow can be assumed. A possible solution is to introduce estimated values for the unknown \( k \)-factor.

Eindhoven, May 1974

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