THE NUMBER OF POLYHEDRA

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1. Introduction

An unsolved problem in the theory of convex polyhedra (and in combinatorics and graph theory as well) is to find a formula giving, or at least an algorithm for calculating, the number of combinatorially distinct polyhedra with a given number of vertices. Grünbaum states, in the 1974 New Encyclopaedia Britannica\(^1\), that "Euler was not successful in his attempts . . . to determine the number of types for each \(v\). Despite efforts of many famous mathematicians since Euler . . . the problem is still open". Shephard\(^{17}\), in his discussion of the unsolved problem, indicates that, failing a solution "it would be a considerable achievement to find a formula which gave a reasonably close approximation . . . when \(v\) is large". The problem is the same if faces are referred to instead of vertices, since one is the dual of the other, and enumeration by the number of edges is in the same category of unsolved problems.

This article presents a compilation of results relating to the above problems. Table I gives the number of polyhedra with \(n\) edges, as far as known, with some

\[
\begin{array}{cccccc}
& 1 & 2 & 3 & 4 & 5 \\
\text{Fig. 1.} & & & & & \\
6 & 7 & 8 & 9 & 10 & \\
\end{array}
\]
estimates for higher $n$, and some further data. Table II gives the number of polyhedra having a given number of faces and vertices for certain ranges of values and estimates for others. The sources of these data and explanations of the tables, and other material, are given in the form of a limited historical review.

This is not the place to define and explain convex polyhedra, symmetry, duality, etc. Combinatorial equivalents here include mirror images, enantiomorphs; two polyhedra are isomorphic, combinatorially equivalent, if their vertices can be placed in one-to-one correspondence such that if two vertices are connected by an edge in one, the corresponding vertices in the other are also connected by an edge.

For illustrative purposes, figs 1 and 2 show the 4-, 5- and 6-faced polyhedra. Figure 1 gives perspective views. Figure 2 gives corresponding projections onto one face from a point outside and close to the center of that face; these are commonly referred to as Schlegel diagrams, although such projections had already been used before Schlegel.

Modern developments in the theory of polyhedra can be said to have commenced with Euler's theorem of 1752 connecting the number of vertices, faces and edges of a polyhedron; however, results relating to the above enumeration problems did not appear until the next century. In 1829, Jacob Steiner (ref. 18, p. 229), in Gergonne's Annales des Mathématiques, listed the 4-, 5- and 6-faced polyhedra according to the number and nature of their faces and asked the question: "What is the general law?" Later, in his book of 1832 (ref. 18, p. 454), he stated that no solution had been forthcoming and rephrased the question, "How many different 7, 8, 9, \ldots, n faced bodies are possible \ldots ?" While Steiner did not so state, he was concerned with combinatorially distinct convex polyhedra. Again no answer was forthcoming.

Considerable work on polyhedra appeared after 1850 but the mathematicians who did the most work on problems of actual enumeration were the Rev.

Fig. 2.
Thomas P. Kirkman (Rector of Croft-with-Southworth) and the two Germans, Professor Oswald Hermes (of Stieglitz) and Professor Max Brückner (of Bautzen).

Attempts at finding a general formula or algorithm for enumerating the classes here considered failed. Kirkman, who initiated the work in a paper of 1855 \[16\] stated in a later paper, of 1878, “Mathematicians will never be satisfied until they can write down the number of $P$-acral $Q$-edra (polyhedra with $P$ vertices and $Q$ faces) in terms of $P$ and $Q$. There is no reason to hope for that achievement from the present power of analysis”. The same thought was echoed many years later by Harary and Palmer in a recent advanced work on enumeration \[14\], where they indicated that the enumeration of 3-connected graphs (see sec. 2) “evidently requires more powerful methods than now exist”.

The results of actual enumeration at the turn of the century are in the works of Hermes and Brückner. These include a list of polyhedra with up to 8 faces \[15\], later verified; a list and diagrams of simple polyhedra with up to 10 faces \[15,6\], later verified; a list of polyhedra with 9 faces and 9 vertices \[15\], later shown to have a duplication and two omissions, and incorrect totals for the number of simple polyhedra with 11 (1250) and 12 faces. These results were obtained by constructing the polyhedra by methods which yielded large masses of figures, which then needed to be compared to eliminate equivalents. Except for some further work by Brückner \[7\] in deriving totals later mainly shown to be incorrect, work in this field practically ceased for a long time. The resumption is considered in the following sections.

2. Planar graphs

The diagrams in fig. 2 can be looked upon in several ways. As networks of points with lines connecting some pairs of them, they are “graphs” of the kind studied in graph theory; they are also in some connections referred to as planar maps. These particular diagrams are “3-connected planar” graphs. A good deal is packed into this term. “Planar” indicates that the graph can be drawn on the surface of a sphere, or in a plane, without any lines crossing each other or meeting, except at the vertices where they are supposed to meet; if it can be so drawn in any manner, it can be done with straight lines, and any planar graph has a dual. “Connected” in general means that the graph is in one piece. The term “3-connected” means that the graph cannot be divided into two parts which are connected to each other at less than three vertices, each of the two parts having at least two edges (a single edge is of course connected to the rest of the graph at only two vertices). This excludes, among others, figures with 2-valent vertices or with faces having only two sides. The diagram of a cube with a V-shaped notch cut into one edge would have a part connected to the remainder at only two vertices. It is not enough that the graph be 3-connected; if two farthest removed diagonally opposite vertices of the skeleton cage of a cube are
joined by a line, the result, considered as a graph, is no longer planar. It is still 3-connected but it cannot be drawn on the surface of a sphere or in a plane without lines crossing or meeting where they should not, and it does not represent a convex polyhedron. The article by Tutte\textsuperscript{25} contains a fuller simple explanation of these concepts.

The vertices and edges of any convex polyhedron form a 3-connected planar graph. The now famous theorem of Steinitz\textsuperscript{20,21}, as restated by Grünbaum (ref. 11, p. 235) is to the effect that any 3-connected planar graph can be realized as a convex polyhedron. Consequently graph theory becomes applicable to polyhedra. This theorem and its consequences seem to have been ignored for some time and do not appear to have been utilized until after 1950. It serves, in some respects, as retroactive validation of some of the things done by the early workers.

The 3-connected planar graphs have turned up in a surprising way in a seemingly entirely unrelated area. This is in connection with the problem of dividing a rectangle into unequal squares. The basic paper of 1940 by Brooks, Smith, Stone and Tutte\textsuperscript{5} showed, by means of an electrical analogy, that any such rectangle (not compounded with another rectangle) can be derived from a 3-connected planar graph, which they then called a c-net, and these were thereafter studied extensively with this application in view. This relationship is referred to by Kac and Ulam in their book “Mathematics and Logic” (Mentor paperback, 1969) as an illustration of “the remarkable and wholly unexpected connections” which occur in mathematics. That polyhedra (in effect) and plane rectangles divided into unequal squares, and Kirchhoff’s laws relating to the flow of electricity in a network, are interrelated, is so intriguing that a digression must be made to show how this is done; not entirely a digression since it was this work which gave the impetus to the revival of the polyhedron enumeration problem. This is done by way of a specific example.

An example, the simplest which gives perfect results, is given in fig. 3, where (a) is the hexahedron number 9 of figs 1 and 2. The edge connecting points a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Fig. 3.}
\end{figure}
and b is removed and the figure is redrawn as in (b), now considered as an electrical network with current I entering at a and leaving at b and the wires each of unit resistance. Current directions are assigned, generally trending downward, but an error is immaterial since if a particular current turns out negative the direction and the sign can be changed simultaneously. The relative values of the currents can be found readily by calling that in wire fc, x, and that in wire df, y, and calculating the values of the others in terms of these (since the resistance of each wire is 1, the voltage between the ends has the same numerical value as the current), which leads to but one final equation, \( y = 7x \). The resultant relative values are written in in (b). The currents and the arrangement of the wires correspond to the 33 × 32 rectangle divided into 9 unequal squares shown at (c). The process can also be regarded as placing an arbitrary e.m.f., a battery, in edge ab of (a) and finding the relative values of the currents in the other edges, considered as wires with unit resistance. Thus you get a rectangle divided into unequal squares from a polyhedron. Placing the battery in edge eb results in a 61 × 69 perfect rectangle. The other edges will result in the same two rectangles, or in an imperfect rectangle, that is, some component squares will be equal, because the polyhedron is self dual, and also has an axis of 180° rotational symmetry.

The theory is developed in the basic paper referred to, and a popular account by Tutte has appeared in the Scientific American 23). A three-part paper by Bouwkamp amplifying and extending the previous results appeared in 1946–47 4).

For use in connection with the generation of rectangles divided into unequal squares, aiming at finding squares divided into unequal squares, sets of c-nets were constructed, first by hand for the lower-order ones, and then by computer. Bouwkamp’s paper gives the diagrams of the c-nets with up to 14 edges, including duals (two omitted ones can be reconstructed from the last line of data on page 71 2)) and Brooks et al. also had constructed these by hand.

Since the interest was focused on the number of edges, a method was needed for producing all the c-nets (3-connected planar graphs) with a given number of edges from those with a lower number of edges. This was accomplished according to a theorem of Tutte developed in graph theory 23), obviously with the objective of its application to squared rectangles. Tutte’s theorem (foreshadowed, in part, by Kirkman, but without actual proof) appears in two forms. According to the first form, all the c-nets with \( n + 1 \) edges can be derived from those with \( n \) edges by performing the following two operations on the latter, in all possible ways: (I) adding a new edge by connecting two non-adjacent vertices of a face, and (II) splitting a vertex having at least 4 incident edges, each of the resultant two vertices having at least two incident edges, and connecting the two vertices with a new edge, and adding in the pyramid with a base of \( (n - 1)/2 \) sides if \( n - 1 \) is even. In performing operation II, the splitting and insertion of
a new edge must be done in such a manner as to avoid losing the planarity of the graph unless non-planar graphs are also desired. The method does not produce pyramids, which must be fed in at the appropriate times. The second form of the theorem, which is limited to planar graphs, dispenses with operation II and dualizes the results of operation I instead; this is the method which was programmed for use with a computer.

In 1960 the Dutch group, Bouwkamp, Duijvestijn and Medema, constructed and tabulated the c-nets with up to 19 edges, but listing only one of a dual pair. This was done by computer, taking nearly 11 hours on the IBM 650 at the Philips Research Laboratories at Eindhoven, The Netherlands. The method and program are given in a thesis by Duijvestijn 8); this involved solution of the problems of the representation of graphs for use by the computer, generating new graphs from old ones, generating dual graphs; and testing the large masses of figures to eliminate duplicates and equivalents. Their work also went on to generate squared rectangles from the c-nets, also by computer. The table of c-nets with up to 19 edges has not been published, but is available 3).

The graphs being equivalent to convex polyhedra, considerable was added to the enumeration of polyhedra by the above work since now in effect all the polyhedra with up to 19 edges had been constructed. And, incidentally, Hermes’ list of 7-hedra and 8-hedra stood checked as the computer produced the same results. Table II, explained in the last section of this article, which lists the number of polyhedra in groups according to the number of faces and vertices, as far as known, has been constructed by utilizing the Bouwkamp et al. table up to 19 edges, as well as other sources.

Diagrams of the c-nets, hence polyhedra, with up to 17 edges have been constructed from the Bouwkamp et al. table and drawn neatly on a set of sheets, by Ray C. Ellis, Jr., of Massachusetts. The table was also used in ref. 9a in checking Hermes’ lists and in ref. 9b, and is referred to in ref. 11.

Simple polyhedra are those with every vertex of degree three. They are the easiest to construct. In 1965 Grace derived and listed the simple polyhedra with up to eleven faces by computer, using the method of dividing faces by a line 10). The program took about 12 hours to run on a Burroughs B 5000 computer at Stanford University. His results confirmed the previous results up to 10 faces but his figure for 11 faces is 1249. Grace used a short-cut in testing for equivalency, adopting a necessary condition of equal surroundedness of corresponding faces which, knowingly, was not a sufficient condition and would introduce errors when \( n \) was very high. A little later Bowen and Fisk generated “triangulations”, polyhedra with all faces triangular, with up to 12 vertices, by computer 4). These are the duals of the simple polyhedra with up to 12 faces and would be the same in number. They obtained 1249 triangulations with 11 vertices, duals of the simple polyhedra with 11 faces, agreeing with Grace, and 7595 triangulations with 12 vertices, duals of the simple polyhedra with 12 faces,
disagreeing with both Hermes (7553) and Brückner (7616). The computing time was 1\(\frac{1}{2}\) hours on the IBM 7094 at the University of California at Berkeley.

3. Rooted planar graphs

The next step to be mentioned and the first step towards a general law is due to Tutte (ref. 24, see ref. 25 for a popular account). Tutte introduced the concept of a “rooted” c-net. One edge is specified as the “root”, with one end as positive and the other end negative, and with the two sides also distinguished, as left and right. On a diagram this can be indicated by an arrow on the root edge and a letter \(l\) and \(r\) on each side. Either side may be marked \(l\) or \(r\) giving two rooted graphs, and the direction of the arrow can be reversed giving two more, making four in all from one edge. Two rooted c-nets which would be equivalent if unrooted, are not equivalent unless they can be brought into coincidence with the rooted edges coinciding as well as their directions and the designations left and right. If the c-net has \(n\) edges, the number of rooted c-nets produced from it will be \(4n\). These will be distinct if the c-net has no element of symmetry. However, if the c-net is symmetrical, the number of distinct rooted c-nets produced from it will be reduced, depending upon the nature of the symmetry. A single plane of symmetry, or a 180° rotational symmetry, will reduce the number to 1/2; two planes of symmetry will reduce the number to 1/4; three planes of symmetry will reduce the number to 1/6 if they pass through one axis and to 1/8 if they are mutually perpendicular. Higher degrees of symmetry will reduce the number still further. Harary and Tutte have shown that the number of rooted c-nets produced from a c-net with \(n\) edges is \(4n/h\), where \(h\) is the order of the automorphism group of the particular graph \(13\).

Tutte has derived an explicit formula giving the number of rooted c-nets for any \(n\). This is the first breakthrough in the problem of finding a general law for the number of polyhedra. He also gives an associated recursion formula used for computations and a table of the number of rooted c-nets up to \(n = 25\). This table is repeated here in table I with some added material. The significance of the achievement of Tutte’s formula may be appreciated when it is considered that there is no formula which enables the number of c-nets to be calculated for a given \(n\), there is also no formula for determining how many of these would be symmetrical, nor any for obtaining the nature of the symmetries, yet here is a formula for calculating a quantity dependent upon these uncalculatable elements.

With the number of rooted c-nets for a given \(n\) known, it is still not yet possible to calculate the number of unrooted c-nets from it, but a fair approximation can be made. As pointed out by Tutte, if it be assumed that the number of symmetrical c-nets becomes negligible in comparison with the number of unsymmetrical ones for large enough \(n\), an approximation of the number of unrooted c-nets can be obtained by dividing the number of rooted ones by \(4n\). This
TABLE I
Number of c-nets

<table>
<thead>
<tr>
<th>Edges</th>
<th>Rooted</th>
<th>Unrooted</th>
<th>Percent Deficiency</th>
<th>Actual Number</th>
<th>Symmetrical</th>
<th>Percent Symmetrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>100.0</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td></td>
<td>4.0</td>
<td>22</td>
<td>16</td>
<td>72.7</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td></td>
<td>2.2</td>
<td>58</td>
<td>32</td>
<td>55.2</td>
</tr>
<tr>
<td>10</td>
<td>24</td>
<td></td>
<td>1.4</td>
<td>158</td>
<td>62</td>
<td>39.2</td>
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<tr>
<td>11</td>
<td>66</td>
<td></td>
<td>1.0</td>
<td>448</td>
<td>123</td>
<td>27.5</td>
</tr>
<tr>
<td>12</td>
<td>214</td>
<td></td>
<td>0.5</td>
<td>1334</td>
<td>234</td>
<td>17.4</td>
</tr>
<tr>
<td>13</td>
<td>676</td>
<td>13</td>
<td>0.9</td>
<td>4199</td>
<td>470</td>
<td>11.2</td>
</tr>
<tr>
<td>14</td>
<td>2209</td>
<td>40</td>
<td>0.6</td>
<td>13384</td>
<td>906</td>
<td>6.8</td>
</tr>
<tr>
<td>15</td>
<td>7296</td>
<td>122</td>
<td>0.5</td>
<td>42767</td>
<td>142948</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>24460</td>
<td>383</td>
<td>0.5</td>
<td>142387</td>
<td>1638248</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>82926</td>
<td>1220</td>
<td>0.5</td>
<td>15364</td>
<td>19328566</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>284068</td>
<td>3946</td>
<td>0.5</td>
<td>67017765</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The approximation is in fact a lower limit. How well it fits is indicated in table I. (Tutte’s formula, when simply divided by 4, also gives an estimate of the number of squared rectangles with \( n - 1 \) component squares.)

In table I the second column gives the number of rooted c-nets as given by Tutte up to \( n = 25 \), with an added line for \( n = 26 \). Column 3 gives the estimates, lower limits in fact, obtained by dividing the numbers in column 2 by 4 times the corresponding value for \( n \); these then are estimates of the number of polyhedra having \( n \) edges. Column 4 gives the percent deficiency of the estimate from the actual numbers as far as known, which are given in column 5. The actual number of c-nets, 3-connected planar graphs, polyhedra, by number of edges is known up to \( n = 19 \) from the Bouwkamp et al. table. As seen from column 4, the deficiency of the estimate becomes less and less with increasing \( n \), going down to 3.5% for \( n = 19 \). Column 6 gives the number of symmetrical c-nets (obtained mainly from the Bouwkamp et al. table) and column 7 the
percentage of symmetrical ones, this percentage decreasing to 6.8 for $n = 19$. The actual number of symmetrical c-nets does not decrease with higher $n$, as seen in column 6; in fact the portion of this column from $n = 13$ to $n = 19$ is very close to a geometric progression with a common ratio of nearly 2. The unsymmetrical ones increase much more rapidly (the numbers, column 5 minus column 6, from $n = 14$ to $n = 19$ are very close to a geometric progression with a common ratio of approximately 3.4), and it is this fact which causes the proportion of symmetrical c-nets to decrease.

Tutte states that the assumption that the number of symmetrical c-nets becomes negligible in proportion for higher $n$ “seems highly plausible to the present author, but no proof of it is known”. The data in the table do not of course constitute a proof of the assumption, but they do increase the plausibility. The Tutte formula, divided by $4n$, is in fact a formula giving an estimate of the number of polyhedra with $n$ edges. The question is whether it is not also an asymptotic formula. This has not been proven but appears to be highly plausible.

That the number of rooted c-nets given by the Tutte formula agrees with the number obtained from the individual polyhedra by the formula $4n/1z$ has been determined up to $n = 17$. The Bouwkamp et al. table identifies the symmetrical c-nets for these and the only problem was determining the order of the automorphism group for them.

4. Number by faces and vertices

The Tutte formula gives only the total number of rooted c-nets for a given number of edges. A further step has been taken by Mullin and Schellenberg who derived a formula for calculating the number of rooted c-nets subdivided according to the number of faces and vertices. The formula is not reproduced here, as neither are the Tutte formulas, as they are quite complicated and would require too much explanation. The paper gives a table of the number of rooted c-nets with up to 16 vertices, subdivided into groups according to the number of faces. To the extent of this table, estimates (lower limits) can be derived for the number of polyhedra having a given number of faces and vertices simply by dividing the number in the table by four times the number of edges. Such estimates are given in table II of this paper.

Table II presents a summary of the number of polyhedra by faces and vertices. Excluding the figures in parentheses, the numbers given in each case save one refer to polyhedra which have been actually derived and listed in published sources or in the Bouwkamp et al. table. The exception is the number for 9 faces and 13 vertices. When a table similar to the present one was first constructed, it was noticed that the column for 9 faces was complete except for one missing group, the second from the bottom. A way was found of separately generating the members of this group, which turned out to be 219, in a simple manner, by applying operation II of Tutte’s theorem in reverse to the individual
members of the last group in the same column 9). These latter were 50 in number and had been listed by Hermes in 1899 and Grace in 1965. The total number of 9-hedra came out as 2606. Hence Steiner's question of 1832 was answered for 9 faces, but it has not yet been answered for his row of three dots nor for his \( n \). The 219 polyhedra with 9 faces and 13 edges have been independently derived by R. M. Foster of New Jersey, and exchange of results showed that the two sets correspond.

The numbers in parentheses in table II are the estimates derived from the Mullin and Schellenberg table, except for the entry for 11 faces and 17 vertices which has been separately calculated in order to make this column complete. In general the actual numbers would not be more than a few percent greater than these estimates. Taking into consideration the groups whose values are known, the estimate (lower limit) for the total number of 10-hedra is 31538 and for the total number of 11-hedra it is 435641. The number of 12-hedra would be considerably over 5000000; extrapolation suggests about 7000000.

The symmetry of table II about the main diagonal from upper left to lower right is due to duality. Groups symmetrically placed with respect to this main diagonal are the same in number of members, which are duals of each other.
The diagonals in the other direction pass through the entries having the same number of edges. The number at the bottom of a column, when the column is complete, is the number of simple polyhedra with the indicated number of faces; the number at the right end of a line, when the line is complete, is the number of polyhedra with all faces triangular. The bottom of the table gives totals and the number of symmetrical polyhedra.

The Mullin and Schellenberg table of rooted c-nets offers a means of checking the number of polyhedra with a given number of faces and vertices from the individual figures of this group if known. This is done by determining the order of the automorphism group for the symmetrical ones and calculating the total number of rooted c-nets. This has been done for the groups with up to 8 faces (therefore also those with up to 8 vertices) and the groups with up to 17 edges, and also the bottom two entries of the column for 9 faces and the bottom entry of the column for 10 faces.

The Mullin and Schellenberg formula, when divided by 4 times the number of edges is a formula giving an estimate of the number of polyhedra having a given number of faces and vertices. The question arises as to whether it is not also an asymptotic formula. For reasons similar to those previously expressed, this conjecture is also quite plausible. This is shown by table III which gives the percentages that the estimates are of the actual numbers where these are known. As can be seen, the percentages increase along the main diagonal and along any

**TABLE III**

Percent closeness of formula estimate

<table>
<thead>
<tr>
<th>vertices</th>
<th>faces</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
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<tbody>
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<tr>
<td>4</td>
<td></td>
<td>4·3</td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td></td>
<td>12·5</td>
<td>8·3</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>8·3</td>
<td>30·0</td>
<td>37·5</td>
<td>13·5</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>7</td>
<td></td>
<td>37·5</td>
<td>49·0</td>
<td>59·1</td>
<td>56·3</td>
<td>22·7</td>
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<td></td>
<td></td>
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<td>80·6</td>
<td>76·9</td>
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<td>9</td>
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<td>80·6</td>
<td>89·6</td>
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<td>91·8</td>
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<td>39·6</td>
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</tr>
<tr>
<td>13</td>
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<td></td>
<td>86·6</td>
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<td>14</td>
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<td>60·2</td>
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<td>16</td>
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<td>75·8</td>
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<td>total</td>
<td></td>
<td>4·3</td>
<td>10·4</td>
<td>24·3</td>
<td>53·1</td>
<td>72·6</td>
<td>89·9</td>
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</table>
diagonal parallel to this one. The lower ones are quite remote, as is common with asymptotic formulas, but the higher ones get closer, in percentages, to the actual numbers.

REFERENCES

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