OPTIMIZATION OF STRICTLY NON-BLOCKING AND REARRANGEABLE INTERCONNECTIONS

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Abstract
The paper is concerned with the study of two-sided connecting networks able to realize any communication state in which an input is connected to at most one output. They play a key role in telephone exchanges and are of growing importance in certain computer systems. Emphasis is put on multi-stage Clos and Karkevitch interconnections in both rearrangeable and strictly non-blocking cases. For each family of interconnections, optimization formulas are derived.

1. Introduction
One of the major problems in the conception of automatic telephone exchanges is the design of switching facilities that perform some given functions and contain a minimum number of crosspoints.

In this paper, we shall be concerned with two-sided connecting networks; they have $N$ inputs, $N'$ outputs, and are able to realize any communication state in which an input is connected to at most one output. Examples of such two-sided networks may be found in the concentration or expansion part of an exchange.

More precisely, we shall study multi-stage interconnections based on the classical three-stage Clos interconnection $^3$: the $(2k + 1)$-stage strictly non-blocking or rearrangeable Clos interconnections, and the $3^k$-stage strictly non-blocking or rearrangeable Karkevitch interconnections. For each of these four families of interconnections, optimization formulas are given (sec. 3). They are based on continuous optimization and give lower bounds rather than precise numerical values.

The last section describes an optimization method that applies to the case of $(2k + 1)$-stage rearrangeable Clos interconnections and yields an exact solution.

Part of the material of sec. 3 is contained in a paper by Bassalygo, Grushko and Neyman $^{11}$, namely the part concerning $(2k + 1)$-stage strictly non-blocking Clos interconnections, in the case where $N = N'$. The method described in sec. 4 is the generalization of an original method due to Benes $^{10}$.)
2. Definitions and basic theorems

The 3-stage Clos interconnection, displayed in fig. 1, is a classical tool in theoretical switching. It consists of three sets of rectangular switches:

(i) \( r \) \((n,m)\)-switches \( S^1_i \) \((i = 1, 2, \ldots, r)\) (1st stage).
(ii) \( m \) \((r,r')\)-switches \( S^2_j \) \((j = 1, 2, \ldots, m)\) (2nd stage).
(iii) \( r' \) \((m,n')\)-switches \( S^3_k \) \((k = 1, 2, \ldots, r')\) (3rd stage).

The \( s \)th input and the \( t \)th output of \( S^i_j \) are denoted \( e^i_{js} \) and \( s^i_{jt} \), respectively.

The interconnection rule, partly sketched in fig. 1, is completely defined by the following equalities:

\[
s^1_{ij} = e^2_{jt}, \tag{1}
\]

\[
s^2_{uv} = e^3_{uv}. \tag{2}
\]

In what follows, we restrict our discussion of network states to situations in which a particular switch input is connected by the switch to at most one switch output. That hypothesis, which excludes some practical applications such as conference telephony, is classical in theoretical switching. Furthermore, it is clear that, under the interconnection rule ((1), (2)) the same hypothesis prevails for the whole network. A connected input–output pair \((e^1_{ij}, s^3_{uv})\) is termed a communication.

There are exactly \( m \) ways to realize a particular communication \((e^1_{ij}, s^3_{uv})\), namely:

\[
e^1_{ij} \rightarrow s^1_{ia} = e^2_{al} \rightarrow s^2_{au} = e^3_{ua} \rightarrow s^3_{uv} \quad (\alpha = 1, 2, \ldots, m). \tag{3}
\]

Each of these \( m \) paths is completely determined by the value of \( \alpha \), i.e. by the index of the 2nd-stage switch it uses.

The matrix model of a Clos interconnection has been introduced by Paull 1) and Neyman 2). The network state is characterized by the connection matrix \( H \). The matrix \( H \) is a rectangular \((r,r')\)-matrix with integer entries. The \((i,j)\)-entry of \( H \), denoted by \( h_{ij} \), is equal to the number of communications relating \( S^i_1 \)

![Fig. 1. Three-stage Clos interconnection.](image-url)
to $S^3_j$. One clearly has
\begin{align}
\sum_{j=1}^{r'} h_{ij} &\leq n \quad (i = 1, 2, \ldots, r), \\
\sum_{j=1}^{r} h_{ij} &\leq n' \quad (j = 1, 2, \ldots, r').
\end{align}

These equations simply state that a switch $S^1_i$ is involved in at most $n$ communications, while a switch $S^3_j$ is involved in at most $n'$ communications. Furthermore, if the pair $(e^1_{11}, s^3_{11})$ is idle, one may write for a similar reason
\begin{align}
\sum_{j=1}^{r'} h_{1j} &\leq n - 1, \\
\sum_{j=1}^{r} h_{1j} &\leq n' - 1.
\end{align}

Similarly, the state of a switch $S^2_k$ in the second stage is characterized by a matrix $H_k$. The matrix $H_k$ is again a rectangular $(r,r')$-matrix but with binary entries. The $(i,j)$-entry of $H_k$, denoted $h^k_{ij}$, is equal to one if the pair of terminals $(e^2_{kl}, s^2_{kj})$ is connected and is equal to zero otherwise. Under our assumptions, we may write
\begin{align}
\sum_{j=1}^{r'} h^k_{ij} &\leq 1 \quad (i = 1, 2, \ldots, r \quad k = 1, 2, \ldots, m), \\
\sum_{i=1}^{r} h^k_{ij} &\leq 1 \quad (j = 1, 2, \ldots, r' \quad k = 1, 2, \ldots, m).
\end{align}

Binary valued matrices satisfying conditions (8) and (9) will be termed bijective matrices.

The matrices $H$ and $H_k$ clearly satisfy the following equality:
\begin{equation}
H = \sum_{k=1}^{m} H_k.
\end{equation}

That equality is important, since it relates the possibility of realizing an arbitrary set of input–output connections described by $H$ to the possibility of representing that matrix as the sum of at most $m$ bijective matrices.

The set
\[ A = \{ H_k \mid k = 1, 2, \ldots, m \} \]
is now partitioned into four pairwise disjoint subsets $A_{ij}$, with $i$ and $j$ in $\{0,1\}$:
\begin{equation}
A_{ij} = \{ H_k \mid \sum_{p=1}^{r'} h^k_{1p} = i \quad \text{and} \quad \sum_{q=1}^{r} h^k_{q1} = j \};
\end{equation}

\[ \sum_{j=1}^{r'} h_{ij} \leq n \quad (i = 1, 2, \ldots, r), \\
\sum_{j=1}^{r} h_{ij} \leq n' \quad (j = 1, 2, \ldots, r').
\]
it is thus the set of matrices $H_k$ with 1 in the first row if $i = 1$ and with 1 in the first column if $j = 1$.

One also defines

$$s_{ij} = \# A_{ij};$$

(12)

it is the number of second-stage switches in which the first input is busy if $i = 1$ and the first output is busy if $j = 1$. The notation $s_{ij}(x)$ designates the value of $s_{ij}$ when the network is in a given state $x$.

The various parameters introduced so far are related by a series of relations that are gathered in the following lemma.

**Lemma 2.1.** (i) $s_{00} + s_{01} + s_{10} + s_{11} = m$. (13)

(ii) If the pair $(e_{11}, s_3)$ is idle then

$$s_{01} + s_{11} \leq n' - 1,$$  

(14)

$$s_{10} + s_{11} \leq n - 1,$$  

(15)

$$s_{01} \leq (r - 1)n,$$  

(16)

$$s_{10} \leq (r' - 1)n',$$  

(17)

$$s_{00} + s_{10} \geq m - n' + 1,$$  

(18)

$$s_{00} + s_{01} \geq m - n + 1,$$  

(19)

$$s_{00} \geq m - r'n' + 1,$$  

(20)

$$s_{00} \geq m - rn + 1.$$  

(21)

**Proof.** (i) Equation (13) simply asserts that the total number of second-stage switches is $m$.

(ii) Equation (14) states that, as $s_{311}$ is idle, the maximum number of communications entering $S^{311}$ is $(n' - 1)$. Similarly, equation (15) states that, as $e_{11}$ is idle, the maximum number of communications originating from $S^{11}$ is $(n - 1)$.

Equation (16) states that the number of communications entering either $S^{32}$ or ... or $S^{3r'}$ may not exceed $(r' - 1)n'$, while equation (17) states that the number of communications originating from either $S^{2}$ or ... or $S^{r}$ may not exceed $(r - 1)n$. 


Relations (18) and (19) are obtained by combining (13), (14) and (15):

\[(13) - (14) \Rightarrow s_{00} + s_{10} \geq m - n' + 1,\]
\[(13) - (15) \Rightarrow s_{00} + s_{01} \geq m - n + 1.\]

Finally

\[(18) - (17) \Rightarrow s_{00} \geq m - r' n' + 1,\]
\[(19) - (16) \Rightarrow s_{00} \geq m - rn + 1.\]

Let us now suppose that the network is in some state $x$. A new call $(e_{11}, s_{3uv})$ may be introduced if some switch $S^2_a$ has both its $i$th input and its $u$th output idle, that is if some matrix $H_a$ has both its $i$th row and its $u$th column identically zero. When no such matrix $H_a$ exists, the state $x$ is termed a blocking state. A network having no blocking states is called strictly non-blocking network.

While examining the possibility of introducing some new call, we are always in position to relabel the switches and their terminals in such a way that the new call becomes $(e_{11}, s_{311})$. The condition under which that new call may be introduced in state $x$ simply becomes $s_{00}(x) \geq 1$.

The essential result is due to Clos 3). The proof of this result essentially follows an argument given by Paull 1).

**THEOREM 2.1.** A three-stage Clos interconnection is strictly non-blocking iff

\[m \geq \min (n + n' - 1, nr, n' r').\]

**Proof.** (1) Sufficiency. Let us suppose that

\[\min (n + n' - 1, nr, n' r') = n + n' - 1.\]

If $s_{00} = 0$ and if $m \geq n + n' - 1$ then, according to (18), $s_{10} \geq n$ and this contradicts (15).

On the other hand, if

\[\min (n + n' - 1, nr, n' r') = nr \text{ or } n' r'\]

then, according to (21) or (20), one deduces that $s_{00} \geq 1$.

(2) Necessity. We first suppose that

\[\min (n + n' - 1, nr, n' r') = n + n' - 1,\]

and that the network is in a state $x$ such that

(i) the pair $(e_{11}, s_{311})$ is idle;
(ii) there are $n' - 1$ communications entering $S^3_1$ and originating from either $S^1_2$ or ... or $S^1_r$; this is possible since $n' - 1 \leq n (r - 1)$;
(iii) there are $n - 1$ communications originating from $S^3_1$ and entering either $S^3_2$ or ... or $S^3_r$; this is possible since $n - 1 \leq n' (r' - 1)$.
In state $x$ one has

$$s_{01}(x) = n' - 1, s_{10}(x) = n - 1 \quad \text{and} \quad s_{01}(x) + s_{10}(x) = n + n' - 2.$$ 

After the introduction of the missing communication $(e^{111}, s^{111})$ one has

$$s_{11} = 1, s_{01} = n' - 1, s_{10} = n - 1 \quad \text{and} \quad s_{11} + s_{01} + s_{10} = n + n' - 1.$$ 

We now suppose that

$$\min (n + n' - 1, n r, n' r') = n r,$$

with

$$r \geq 2 \quad \text{and} \quad r' \geq 2,$$

and that the network is in a state $x$ such that

(i) the pair $(e^{111}, s^{311})$ is idle;
(ii) there are $n (r - 1)$ communications entering $S^3_1$ and originating from either $S^1_2$ or ... or $S^1_r$; this is possible since $n (r - 1) \leq n' - 1$;
(iii) there are $n - 1$ communications originating from $S_{11}$ and entering either $S^3_2$ or ... or $S^3_{r'}$; this is possible since $n - 1 \leq n' (r' - 1)$.

In state $x$ one has

$$s_{01}(x) = n (r - 1), s_{10}(x) = n - 1 \quad \text{and} \quad s_{01}(x) + s_{10}(x) = n r - 1.$$ 

After the introduction of the missing communication $(e^{111}, s^{111})$

$$s_{11} + s_{01} + s_{10} = n r.$$ 

If $r = 1$ or if $r' = 1$, every set of $p$ communications uses $p$ different second-stage switches, and thus

$$m \geq \min \{n r, n' r'\}.$$ 

Finally, the case where

$$\min (n + n' - 1, n r, n' r') = n' r'$$

may be treated in a similar way. \qed

Let us now consider a network that may contain blocking states but has the following property: every new call may be introduced under the condition that in some cases the route followed by some calls already in progress has to be modified. Such networks are called rearrangeable networks.

The following theorem generalizes a result known as the Slepian–Duguid theorem (4,5) that was stated in the symmetrical case i.e. when $n = n'$, $r = r'$.

The proof follows a constructive method due to Paull (1).

**THEOREM 2.2.** A three-stage Clos interconnection is rearrangeable iff

$$m \geq \min [\max (n, n'), n r, n' r'].$$


Proof. (1) Necessity. Let us suppose that

\[ \min \{ \max (n, n'), nr, n' r' \} = \max (n, n') = n, \]

and consider a state in which there are \( n \) communications originating from \( S_1 \); this is possible since \( n \leq n' r' \). These \( n \) communications use \( n \) distinct second-stage switches and thus \( m \geq n \).

If

\[ \min \{ \max (n, n'), nr, n' r' \} = nr \]

then either

\[ \max (n, n') = n \quad \text{and} \quad r = 1, \]

or

\[ \max (n, n') = n' \quad \text{and} \quad nr \leq n'. \]

In the first case there is a state in which \( n \) communications originate from \( S_1 \); thus \( m \geq n = nr \).

In the second case, there is a state in which \( nr \) communications enter \( S_3 \); thus \( m \geq nr \).

(2) Sufficiency. If

\[ \min \{ \max (n, n'), nr, n' r' \} = nr \]

and if

\[ m \geq nr \]

the interconnection is strictly non-blocking: this is a consequence of theorem 2.1 and of the fact that \( \max (n, n') \leq n + n' - 1 \).

It remains to examine the case where

\[ \min \{ \max (n, n'), nr, n' r' \} = \max (n, n'). \]

As above, we suppose that the network is in a state \( x \) in which the pair \((e^{11}, s^{31})\) is idle. According to the inequalities (18) and (19) one observes that either

\[ s_{00}(x) \geq 1 \]

or

\[ s_{01}(x) \geq 1 \quad \text{and} \quad s_{10}(x) \geq 1. \]

If \( s_{00}(x) \geq 1 \), no rearrangement is necessary in order to introduce the communication \((e^{11}, s^{31})\). If \( s_{00}(x) = 0 \), let us choose a matrix \( H_k \) in \( A_{01} \) and a matrix \( H_l \) in \( A_{10} \). It is possible to build up two sequences of 1 entries selected alternatively in \( H_k \) and \( H_l \):

\[ S_A = h_{p_{11}}, h_{p_{12}}, h_{t_{11}}, h_{l_{11}} h_{l_{12}}, h_{l_{22}}, \ldots, \]

\[ S_B = h_{l_{11}}, h_{l_{12}}, h_{l_{21}}, h_{l_{22}} h_{l_{23}}, h_{l_{33}}, \ldots. \]

These sequences are uniquely determined and are furthermore disjoint: indeed,
the identity of two members of these sequences like e.g. \( h^t_{1,1} \) and \( h^t_{2,2} \) implies \( i_1 = s_2 \) and thus, due to the bijective character of \( H_k \), this also implies the identity of \( h^k_{1,1} \) and \( h^k_{2,2} \). Iteration of that process would imply a contradiction such as the identity of \( h^k_{p,1} \) and \( h^k_{s,1} \). Now, if one selects one of the two sequences, and decides to route over \( S^2_k \) all the communications appearing in that sequence and initially routed over \( S^2_1 \), and conversely, one obtains a new network state that is equivalent to the former state and non-blocking for \( \{ e_{1,1}, s_{3,1} \} \). The number of talks to be moved is the length of the shortest of \( S_A \) and \( S_B \).

An important definition that must be recalled is that of crosspoint. Indeed, the various switches that are the building blocks of the Clos interconnections are made up of regularly disposed contacts located at the crosspoints of the input and output lines. A \((m,n)\)-switch thus contains \( mn \) crosspoints and this number is a measure of its cost, whatever the used technology may be (electromechanical devices, electronics ...).

The number of crosspoints in the Clos interconnection of fig. 1 is

\[
rmn + mrr' + r'mn' = m \left( N + N' + \frac{NN'}{nn'} \right),
\]

where \( N = nr \) is the total number of inputs, and \( N' = n'r' \) the total number of outputs.

The interest of the Clos interconnection is in obtaining a network that has the same performances as a \((N, N')\)-switch, but contains fewer crosspoints. In figure 1 one could replace the \( m \) second-stages switches by \( m \) identical strictly non-blocking (rearrangeable) three-stage Clos interconnections; one obtains in this way a strictly non-blocking (rearrangeable) 5-stage Clos interconnection. By iterating the process one may define Clos networks with 7, 9, ..., \( 2k + 1 \) stages.

Being given the number of inputs \( N \) and the number of outputs \( N' \), a 3-stage Clos interconnection is entirely characterized by the three parameters \( n, n' \) and \( m \), since \( r = N/n \) and \( r' = N'/n' \).

More generally, a \((2k + 1)\)-stage Clos interconnection is characterized by the \( 3k \) parameters

\[ n_i, n'_i \text{ and } m_i \quad (i = 1, \ldots, k). \]

It is fully described by the following theorem.

**Theorem 2.3.** Consider a \((2k + 1)\)-stage Clos interconnection, with \( N \) inputs and \( N' \) outputs, characterized by the \( 3k \) parameters \( n_i, n'_i \) and \( m_i \), \( i = 1, \ldots, k \). The following assertions hold true:

(i) The products \( n_1 \ldots n_k \) and \( n'_1 \ldots n'_k \) divide \( N \) and \( N' \), respectively.
(ii) If it is strictly non-blocking then

\[ m_i \geq \min \left( \frac{N}{n_i \ldots n_{i-1}}, \frac{N'}{n'_{1} \ldots n'_{i-1}} \right), \]  

for every \( i = 1, \ldots, k \); if it is rearrangeable then

\[ m_i \geq \min \left( \max (n_i, n'_i), \frac{N}{n_1 \ldots n_{i-1}}, \frac{N'}{n'_{1} \ldots n'_{i-1}} \right), \]  

for every \( i = 1, \ldots, k \).

(iii) There are \( \frac{m_1 \ldots m_{i-1} N}{n_1 \ldots n_i} \) switches of type \((n_i, m_i)\) in stage \( i \), for \( i = 1, \ldots, k \); there are \( m_1 \ldots m_k \) switches of type \((\frac{N}{n_1 \ldots n_k}, \frac{N'}{n'_{1} \ldots n'_{k}})\) in stage \( k + 1 \); there are \( \frac{m_1 \ldots m_{i-1} N'}{n'_{1} \ldots n'_i} \) switches of type \((m_i, n'_i)\) in stage \( 2k + 2 - i \), for \( i = 1, \ldots, k \).

(iv) The interconnection contains

\[ P = \frac{m_1 \ldots m_k N N'}{n_1 \ldots n_k n'_1 \ldots n'_k} + \sum_{i=1}^{k} m_1 \ldots m_i \left( \frac{N}{n_1 \ldots n_{i-1}} + \frac{N'}{n'_{1} \ldots n'_{i-1}} \right) \]  

crosspoints.

Proof. In what concerns points (i) to (iii), the proof is by induction. Indeed, for \( k = 1 \), one has a 3-stage Clos interconnection that contains

\[ r = \frac{N}{n} \]  
\((n, m)\)-switches in the first stage,

\[ m \left( r = \frac{N}{n}, r' = \frac{N'}{n'} \right) \]  
-switches in the second stage, and

\[ r' = \frac{N'}{n'} \]  
\((m, n')\)-switches in the third stage.

It is obvious that \( n \) divides \( N \) and that \( n' \) divides \( N' \). Furthermore for a strictly non-blocking interconnection one has

\[ m \geq \min (n + n' - 1, N, N'), \]

and for a rearrangeable interconnection

\[ m \geq \min \left[ \max (n, n'), N, N' \right]. \]

Let us now consider a \((2k - 1)\)-stage interconnection, with parameters \( n_i \), \( n'_i \), and \( m_i \), \( i = 1, \ldots, k - 1 \), and having properties (i) to (iii).
One replaces each of the $m_1 \ldots m_{k-1}$ switches of type
\[
\left( \frac{N}{n_1 \ldots n_{k-1}}, \frac{N'}{n'_1 \ldots n'_{k-1}} \right)
\]
of the central stage by a 3-stage interconnection with parameters $n_k, n'_k$ and $m_k$.

This construction does not modify stages 1 to $k-1$, and changes the stage number $2(k-1) + 2 - i$ into number $2k + 2 - i$, for $i = 1, \ldots, k-1$. It remains to verify the theorem for stages number $k$, $k + 1$ and $k + 2$.

Let us examine one of the 3-stage interconnections, with parameters $n_k, n'_k$ and $m_k$, that replaces one of the $m_1 \ldots m_{k-1}$ switches of type
\[
\left( \frac{N}{n_1 \ldots n_{k-1}}, \frac{N'}{n'_1 \ldots n'_{k-1}} \right);
\]
$n_k$ divides $\frac{N}{n_1 \ldots n_{k-1}}$, $n'_k$ divides $\frac{N'}{n'_1 \ldots n'_{k-1}}$;
according to theorems 2.1 and 2.2

\[
m_k \geq \min \left( n_k + n'_k - 1, \frac{N}{n_1 \ldots n_{k-1}}, \frac{N'}{n'_1 \ldots n'_{k-1}} \right)
\]
or
\[
m_k \geq \min \left[ \max (n_k, n'_k), \frac{N}{n_1 \ldots n_{k-1}}, \frac{N'}{n'_1 \ldots n'_{k-1}} \right].
\]

Every 3-stage interconnection contains
\[
\frac{N}{n_1 \ldots n_{k-1} n_k} \text{ switches of type } (n_k, m_k),
\]
\[
m_k \frac{N}{n_1 \ldots n_k} \text{ switches of type } \left( \frac{N}{n_1 \ldots n_k}, \frac{N'}{n'_1 \ldots n'_{k}} \right) \text{ and}
\]
\[
\frac{N'}{n'_1 \ldots n'_{k-1} n'_k} \text{ switches of type } (m_k, n'_k).
\]

By multiplying these numbers by $m_1 \ldots m_{k-1}$ one obtains the announced result.

Point (iv) is a direct consequence of point (iii).

We are now in place to state one of the problems we shall be dealing with in the sequel of this paper: being given the positive integers $N$ and $N'$, design a $(2k + 1)$-stage Clos interconnection with $N$ inputs, $N'$ outputs and containing a minimum number of crosspoints.
LEMMA 2.2. If we are looking for a Clos interconnection with a minimum number of crosspoints, then conditions (23) and (24) may be replaced by

\[ m_t \geq n_t + n'_t - 1, \] (26)

and

\[ m_t \geq \max(n_t, n'_t), \] (27)

respectively.

Proof. Let us suppose that for some \( j \) in \( \{1, \ldots, k\} \) condition (23) or (24) becomes

\[ m_j \geq \frac{N}{n_1 \ldots n_{j-1}}. \] (28)

Then, if \( j \geq 2 \), we may write the following series of inequalities in which \( P \) is defined by (25):

\[
P \geq J-1 \sum_{t=1}^{J-1} m_1 \ldots m_t \left( \frac{N}{n_1 \ldots n_{t-1}} + \frac{N'}{n'_1 \ldots n'_{t-1}} \right) + m_1 \ldots m_J \frac{N'}{n'_1 \ldots n'_{J-1}}.
\]

This proves that the \( (2k + 1) \)-stage interconnection does not contain a minimum number of crosspoints.

If \( j = 1 \), we may write

\[ P \geq m_1 N' \geq N N', \]

and obtain the same conclusion.

LEMMA 2.3. If we are looking for a Clos interconnection with a minimum number of crosspoints, we may add the following conditions:

(i) \( n_t \geq 2 \),

(ii) \( n'_t \geq 2 \),

(iii) \( n_1 \ldots n_k < N \),

(iv) \( n'_1 \ldots n'_k < N' \),

for every \( i = 1, \ldots, k \).

Proof. (i) Let us suppose that for some \( j \) in \( \{1, \ldots, k\} \) one has \( n_j = 1 \). On the other hand, notice that

\[ m_j \geq \max(n_j, n'_j) \geq n'_j. \]
If \( k \geq 2 \) we may write the following inequality in which \( P \) is defined by (25):

\[
P > \frac{m_1 \ldots m_{j-1} m_{j+1} \ldots m_k N N'}{n_1 \ldots n_{j-1} n_{j+1} n_k n'_1 \ldots n'_{j-1} n'_{j+1} \ldots n'_k}
+ \sum_{i=1}^{j-1} m_1 \ldots m_i \left( \frac{N}{n_1 \ldots n_{i-1}} + \frac{N'}{n'_1 \ldots n'_{i-1}} \right)
+ \sum_{i=j+1}^{k} m_1 \ldots m_{j-1} m_{j+1} \ldots m_i \left( \frac{Nn_j}{n_1 \ldots n_{i-1}} + \frac{N' n'_j}{n'_1 \ldots n'_{i-1}} \right).
\]

If \( k = 1 \) one obtains

\[
P > \frac{m_1 N N'}{n_1 n'_1} \geq NN'.
\]

In both cases one deduces that the \((2k + 1)\)-stage interconnection does not contain a minimum number of crosspoints.

(ii) Same proof.

(iii) Let us suppose that \( n_1 \ldots n_k = N \). Therefore

\[
m_k \geq \max(n_k, n'_k) \geq n_k = \frac{N}{n_1 \ldots n_{k-1}}.
\]

If \( k \geq 2 \) we may write

\[
P > \sum_{i=1}^{k-1} m_1 \ldots m_i \left( \frac{N}{n_1 \ldots n_{i-1}} + \frac{N'}{n'_1 \ldots n'_{i-1}} \right) + m_1 \ldots m_k \frac{N'}{n'_1 \ldots n'_{k-1}}
> \frac{m_1 \ldots m_{k-1} N N'}{n_1 \ldots n_{k-1} n'_1 \ldots n'_{k-1}} + \sum_{i=1}^{k-1} m_1 \ldots m_i \left( \frac{N}{n_1 \ldots n_{i-1}} + \frac{N'}{n'_1 \ldots n'_{i-1}} \right).
\]

If \( k = 1 \)

\[
P > m_1 N' \geq NN'.
\]

(iv) Same proof.

\[\square\]

In modern technology, switches are mostly realized with the numbers of inputs and outputs equal to powers of 2. In that case we may suppose that

\[
n_i = 2^{v_i}, n'_i = 2^{v'_i}, m_i = 2^{u_i} \quad (i = 1, \ldots, k),
\]

\[
\frac{N}{2^{v_1 + \ldots + v_k}} = 2^e, \quad \frac{N'}{2^{v'_1 + \ldots + v'_k}} = 2^{e'}
\]
and thus

\[ N = 2^r, \quad N' = 2^{r'}. \]

**THEOREM 2.4.** In a Clos interconnection with a minimum number of cross-points, composed of switches the numbers of inputs and outputs of which are equal to powers of 2, the following properties are verified:

(i) \( v_i = v'_i \),

(ii) \( \mu_i = v_i + 1 \) in the strictly non-blocking case,

\( \mu_i = v_i \) in the rearrangeable case.

**Proof.** According to (26) and (27) one must have

\[ 2^{m_i} \geq 2^{v_i} + 2^{v'_i} - 1 \quad \text{(strictly non-blocking case)}, \]

or

\[ 2^{m_i} \geq \max (2^{v_i}, 2^{v'_i}) \quad \text{(rearrangeable case)}. \]

From (33), (29) and (30) one deduces that

\[ \mu_i \geq \max (v_i, v'_i) + 1. \]

From (34)

\[ \mu_i \geq \max (v_i, v'_i). \]

Furthermore, if the number of crosspoints is minimum one has

\[ \mu_i = \max (v_i, v'_i) + 1, \]

in the strictly non-blocking case, and

\[ \mu_i = \max (v_i, v'_i), \]

in the rearrangeable case.

Let us now suppose that for a given index \( j \) one has \( v_j \leq v'_j \) and thus either

\[ \mu_j = v'_j + 1 \quad \text{or} \quad \mu_j = v'_j. \]

If the number of crosspoints is minimum, then \( v_j \) must be as great as possible while being not greater than \( v'_j \) and not greater than

\[ v - (v_1 + \ldots + v_{j-1} + v_{j+1} + \ldots + v_k), \]

that is either

\[ v_j = v'_j \quad \text{or} \quad v_j = v - (v_1 + \ldots + v_{j-1} + v_{j+1} + \ldots + v_k); \]

however, the second equality may not hold true as proven by the inequality (31). \( \square \)

In the sequel of this paper, we shall only consider \((2k+1)\)-stage Clos interconnections in which

\[ n_i = n'_i, \]

\[ m_i = 2n_i \quad \text{(strictly non-blocking case)}, \]

\[ m_i = n_i \quad \text{(rearrangeable case)}, \]
for every $i = 1, \ldots, k$. As proven by the preceding theorem, this is not a restriction in the practical cases where all parameters are powers of two.

The numbers of crosspoints then become

$$P = \frac{2^k NN'}{n_1 \ldots n_k} + (N + N') \sum_{i=1}^{k} 2^i n_i \quad (\text{strictly non-blocking case}), \quad (37)$$

and

$$P = \frac{NN'}{n_1 \ldots n_k} + (N + N') \sum_{i=1}^{k} n_i \quad (\text{rearrangeable case}). \quad (38)$$

Another construction yielding strictly non-blocking or rearrangeable networks is due to Karkevitch \(^6\) and Liénard \(^7\): in figure 1 one may replace the $r$ first-stage switches by $r$ identical strictly non-blocking (rearrangeable) three-stage Clos interconnections, the $m$ second-stage switches by $m$ identical s.n.b. (r.) three-stage C.i., and the $r'$ third-stage switches by as many identical s.n.b. (r.) three-stage C.i. One obtains in this way a 9-stage strictly non-blocking (rearrangeable) interconnection. By iterating the process one may define Clos networks with $27, 81, \ldots, 3^k$ stages.

In order to avoid confusion between $(2k + 1)$- and $3^k$-stage networks, the first ones will be referred to as Clos interconnections, while the second ones will be called Karkevitch interconnections.

Davio \(^8\) has stated a theorem similar to theorem 2.3 that fully describes the $3^k$-stage Karkevitch interconnection. However, its presentation needs the introduction of several new notations and its proof is rather long; this theorem and its consequences could be the subject of a future paper. For these reasons, we give a minus detailed but simpler description of these networks. A $3^k$-stage Karkevitch interconnection, displayed in fig. 2, consists of three sets of $3^{k-1}$-stage Karkevitch interconnections:

(i) $R_k$ interconnections, with $N_k$ inputs and $M_k$ outputs;
(ii) $M_k$ interconnections, with $R_k$ inputs and $R'_k$ outputs;
(iii) $R'_k$ interconnections, with $M_k$ inputs and $N'_k$ outputs.

If the total number of inputs and outputs are $N$ and $N'$, respectively, then

$$N = N_k R_k \quad \text{and} \quad N' = N'_k R'_k.$$

As in the case of Clos interconnections, we restrict our attention to the case where

$$N_k = N'_k, \quad M_k = 2N_k \quad (\text{strictly non-blocking case}),$$
$$M_k = N_k \quad (\text{rearrangeable case}).$$
Let us now denote by $P_1(k-1)$, $P_2(k-1)$ and $P_3(k-1)$ the number of crosspoints in every $3^k-1$-stage Karkevitch interconnection of the first second and third set. If we denote by $P(k)$ the number of crosspoints in the entire $3^k$-stage interconnection, we may write

$$P(k) = \frac{N}{N_k} P_1(k - 1) + M_k P_2(k - 1) + \frac{N'}{N_k} P_3(k - 1).$$  

(39)

3. Continuous optimization. Asymptotic behaviour

The main goal we are pursuing is to design a multi-stage interconnection, with given numbers $N$ of inputs, $N'$ of outputs, that contains a minimum number of crosspoints.

For that purpose we must choose a type of interconnection and then search for suitable values of the parameters. Therefore, the functions we are dealing with in this optimization problem are integer functions of integer variables, that is discrete functions.

However, in this section we shall handle these functions as continuous ones; so, by using classical derivation methods, we shall obtain results that are not entirely correct from a numerical point of view, but give a very good idea of the way the various parameters may vary with $N$ and $N'$. This allows to slightly modify $N$ and $N'$ in order to minimize the number of crosspoints.

3.1. $(2k + 1)$-stage Clos interconnections

Let us consider that (37) and (38) define two real functions of $k + 1$ real variables $n_1, \ldots, n_k, k$.

We put

$$Q = \frac{2 NN'}{N + N'}.$$  

(40)
THEOREM 3.1. The minimum number of crosspoints for a \((2k + 1)\)-stage strictly non-blocking Clos interconnection is obtained for
\[
n_i = f_i(k) \, Q^{1/(k+1)} \, 2^{1-i} \quad (i = 1, \ldots, k),
\]
where
\[
f_i(k) = 2^{\frac{k^2 + k - 4}{(k+1)}},
\]
and is given by
\[
P_{opt}(k) = NN' \, g_1(k) \, Q^{-k/(k+1)},
\]
where
\[
g_1(k) = (1 + k) \, 2^{\frac{k^2 + 1}{(k+1)}}.
\]
The optimum number of stages corresponds to
\[
k = -(1 + 1/\ln 2) + [(1/\ln 2)^2 + 2 \log_2 Q - 4]^{\frac{1}{2}}
\]
if \(\log_2 Q \geq \frac{1}{2} + 1/\ln 2\), and corresponds to \(k = 0\) in the contrary case.

Proof. Let us study the continuous function \(p\) defined as follows:
\[
p(n_1, \ldots, n_k) = \frac{2k \, NN'}{n_1 \cdots n_k} + (N + N') \sum_{i=1}^{k} 2^i \, n_i,
\]
and thus
\[
\frac{\partial p}{\partial n_i} = -\frac{2^k \, NN'}{n_1 \cdots n_i^2 \cdots n_k} + 2^i(N + N') \quad \text{for every} \quad i = 1, \ldots, k.
\]
We are looking for a point \((n_1, \ldots, n_k) > 0\), that is \(n_i > 0\) for every \(i = 1, \ldots, k\), where \(p\) is minimum.

The \(k\) partial derivatives \(\frac{\partial p}{\partial n_i}\) are equal to zero at point \((n_1, \ldots, n_k) > 0\) iff
\[
n_1 \cdots n_i^2 \cdots n_k = 2^{k-i-1} \, Q \quad (i = 1, \ldots, k)
\]
in particular
\[
n_1^2 \cdots n_k = 2^{k-2} \, Q.
\]
By dividing the two former equalities, one obtains
\[
n_i = 2^{1-i} \, n_{i-1} \quad (i = 1, \ldots, k).
\]
One then deduces
\[
n_1^{k+1} \, 2^{1-(2+k)} = 2^{k-2} \, Q
\]
and thus
\[
n_1^{k+1} = 2^{(k^2 + k - 4)/2} \, Q.
\]
Therefore, since \( n_1 > 0 \), one obtains

\[
n_1 = f_1(k) \, Q^{1/(k+1)},
\]

and

\[
n_i = f_1(k) \, Q^{1/(k+1)} \, 2^{i-1}, \quad (i = 1, \ldots, k).
\]

By introducing these values in \( p(n_1, \ldots, n_k) \) one obtains (43).

In order to compute the best value for \( k \), let us define the continuous functions \( h \) and \( h' \) as follows:

\[
h(k) = g_1(k) \, Q^{-k/(k+1)},
\]

\[
h'(k) = \log_2 h(k) = \log_2 (1 + k) + \frac{k(k + 5)}{2(k + 1)} - \frac{k}{k + 1} \log_2 Q.
\]

We are looking for a point \( k \geq 0 \) where \( h \), and thus also \( h' \) are minimum:

\[
\frac{dh'}{dk} = \frac{1}{(1 + k) \ln 2} + \frac{k^2 + 2k + 5}{2(k + 1)^2} \frac{\log_2 Q}{(k + 1)^2}.
\]

The equation \( \frac{dh'}{dk} = 0 \), with \( k \geq 0 \), is equivalent to

\[
2(k + 1)^2 \frac{dh'}{dk} = k^2 + (2/\ln 2 + 2) k + (2/\ln 2 + 5 - 2 \log_2 Q) = 0,
\]

the two roots of which are

\[
k = -(1 + 1/\ln 2) \pm [(1/\ln 2)^2 + 2 \log_2 Q - 4]^{1/2}.
\]

If

\[
\log_2 Q \geq \frac{5}{2} + 1/\ln 2 \approx 15.4,
\]

the greatest root is nonnegative. In the contrary case,

\[
2(k + 1)^2 \frac{dh'}{dk} > 0 \quad (\forall k \geq 0),
\]

and thus

\[
\frac{dh'}{dk} > 0 \quad (\forall k \geq 0).
\]

The minimum of \( h' \) corresponds to \( k = 0 \). \( \square \)

A similar reasoning allows to prove the following theorem.

**Theorem 3.2.** The minimum number of crosspoints for a \((2k + 1)\)-stage rearrangeable Clos interconnection is obtained for

\[
n_i = f_2(k) \, Q^{1/(k+1)} \quad (i = 1, \ldots, k),
\]

(46)
where
\[ f_2(k) = 2^{-1/(k+1)}, \] (47)

and is given by
\[ P_{\text{opt}}(k) = NN' g_2(k) Q^{-k/(k+1)}, \] (48)

where
\[ g_2(k) = (1 + k) 2^{k/(k+1)}, \] (49)
The optimum number of stages corresponds to
\[ k = \ln \frac{Q}{2} - 1 \] (50)
if \( Q \geq 2e \), and corresponds to \( k = 0 \) in the contrary case.

3.2. 3\(^k\)-stage Karkevitch interconnection

We state two theorems, similar to theorems 3.1 and 3.2, concerning the Karkevitch interconnections.

Notice that for \( k = 1 \), there is no difference between Clos and Karkevitch interconnections; therefore, theorems 3.1 and 3.2, with \( k = 1 \), may be used as starting points of a recurrence proof.

THEOREM 3.3. The minimum number of crosspoints for a 3\(^k\)-stage strictly non-blocking Karkevitch interconnection is obtained for
\[ N_k = f_3(k) Q^k, \] (51)

where
\[ f_3(k) = \frac{1}{2} (\sqrt{3})^{2^k-2}, \] (52)

and is given by
\[ P_{\text{opt}}(k) = NN' g_3(k) Q^{-1+2^{-k}} \] (53)

where
\[ g_3(k) = \frac{4}{3} (2 / \sqrt{6})^k (\sqrt{3})^{2^{-k}}. \] (54)
The optimum number of stages corresponds to
\[ k = \log_2 \left[ \frac{\ln (3Q/4) \ln 2}{\ln (2 / \sqrt{6})} \right] \] (55)
if this value is positive; it corresponds to \( k = 0 \) in the contrary case.
Proof. By induction on \( k \). For \( k = 1 \), one deduces from theorem 3.1 that

\[
N_1 = f_1(1) Q^+ = 2^{-\frac{1}{2}} Q^+ = f_3(1) Q^+,
\]
\[
P_{opt}(1) = NN' g_1(1) Q^{-\frac{1}{2}} = NN' 2^{q_\infty} Q^{-\frac{1}{2}} = NN' g_3(1) Q^{-\frac{1}{2}}.
\]

Let us suppose that

\[
P_{opt}(k - 1) = NN' g_3(k - 1) Q^{-1 + 2^{-k+1}}.
\]

We may compute \( P_{opt}(k) \) by replacing, in (39), \( P_1(k - 1) \), \( P_2(k - 1) \) and \( P_3(k - 1) \) by their minimum value, and by minimizing \( P(k) \):

\[
P_1(k - 1) = N_k M_k g_3(k - 1) \left( \frac{2 N_k M_k}{N_k + M_k} \right)^{-1 + 2^{-k+1}}
\]

and thus, since \( M_k = 2N_k \),

\[
P_1(k - 1) = 2 N_k^2 g_3(k - 1) \left( \frac{4}{3} N_k \right)^{-1 + 2^{-k+1}},
\]
\[
P_2(k - 1) = NN' \quad g_3(k - 1) \left( \frac{Q}{N_k} \right)^{-1 + 2^{-k+1}},
\]

and

\[
P_3(k - 1) = P_1(k - 1).
\]

Therefore, according to (39),

\[
P(k) = 2(N + N') N_k g_3(k - 1) \left( \frac{4}{3} N_k \right)^{-1 + 2^{-k+1}} + 2 \frac{NN'}{N_k} g_3(k - 1) \left( \frac{Q}{N_k} \right)^{-1 + 2^{-k+1}}
\]
\[
= (N + N') g_3(k - 1) \left[ 2 \left( \frac{4}{3} \right)^{-1 + 2^{-k+1}} N_k^{2^{-k+1}} + Q^{2^{-k+1}} N_k^{-2^{-k+1}} \right]. \tag{56}
\]

By derivating with respect to \( N_k \) one obtains

\[
\frac{\partial P(k)}{\partial N_k} = (N + N') g_3(k - 1)
\]
\[
\times [2 \left( \frac{4}{3} \right)^{-1 + 2^{-k+1}} N_k^{2^{-k+1}-1} Q^{2^{-k+1}} N_k^{-2^{-k+1} - 1}];
\]

this partial derivative is equal to zero iff

\[
N_k^{2^{-k+2}} = Q^{2^{-k+1}} \left( \frac{4}{3} \right)^{1 - 2^{-k+1}}
\]

that is

\[
N_k = \frac{1}{2} \left( \sqrt{3} \right) \left( \frac{4}{3} \right)^{2^{-k}-2} (\sqrt{Q}). \tag{57}
\]

The introduction of (57) in (56) yields (53).
The optimum value of $k$ is next derived from the equation

$$\frac{d \ln P_{opt}(k)}{dk} = 0$$

that is equivalent to the equation

$$2^k = \frac{\ln (3Q/4) \ln 2}{\ln (2 \sqrt[3]{6})}.$$

A very similar reasoning allows to prove the following theorem.

THEOREM 3.4. The minimum number of crosspoints for a $3^k$-stage rearrangeable Karkevitch interconnection is obtained for

$$N_k = f_4(k) \ Q^k$$

where

$$f_4(k) = 2^{-2k^2}.$$  \hspace{1cm} (59)

and is given by

$$P_{opt}(k) = NN' \ g_4(k) \ Q^{-1 + 2^{-k}},$$

where

$$g_4(k) = (2 \sqrt[3]{2})^k.$$  \hspace{1cm} (61)

The optimum number of stages corresponds to

$$k = \log_2 \left[ \frac{\ln Q \ln 2}{\ln (2 \sqrt[3]{2})} \right]$$

if this value is positive, and to zero in the contrary case.  \hspace{1cm} (62)

3.3. Comparison of the different structures

By using the relations (43), (48), (53) and (60) we may compare the various types of interconnections described above.

3.3.1. Strictly non-blocking interconnections

In figure 3, the reduced number of crosspoints $P_{opt}(k)$ is given for different structures:

1) the $(N,N')$-switch;
2) the 3-stage Clos interconnection;
3) the 5-stage Clos interconnection;
4) the 9-stage Karkevitch interconnection.
Furthermore, in case 2, instead of using (43) with $k = 1$, we use the following formula:

$$P_{\text{opt}} = NN' g_1(1) Q^{-\frac{1}{2}} [1 - (2 Q)^{-\frac{1}{2}}].$$

It has been established by Liénard \cite{7} by putting $m = 2n - 1$ rather than $m = 2n$.

The best choice among these structures is:
- $Q \leq 23$ the simple switch;
- $23 < Q \leq 447$ the 3-stage Clos interconnection;
- $447 < Q \leq 3409$ the 5-stage Clos interconnection;
- $3409 < Q \leq 442368$ the 9-stage Karkevitch interconnection.

### 3.3.2. Rearrangeable interconnections

The reduced numbers of crosspoints are given in fig. 4, for different structures:

1. the $(N,N')$-switch;
2. the 3-stage Clos interconnection;
3. the 5-stage Clos interconnection;
4. the 7-stage Clos interconnection;
5. the 9-stage Clos interconnection;
6. the 11-stage Clos interconnection;
7. the 13-stage Clos interconnection;
8. the 9-stage Karkevitch interconnection;
9. the 27-stage Karkevitch interconnection.
The best choice among these structures is thus

- $Q \leq 8$ the simple switch;
- $8 < Q \leq 22$ the 3-stage Clos interconnection;
- $22 < Q \leq 63$ the 5-stage Clos interconnection;
- $63 < Q \leq 173$ the 7-stage Clos interconnection;
- $173 < Q \leq 474$ the 9-stage Clos interconnection;
- $474 < Q \leq 1296$ the 11-stage Clos interconnection;
- $1296 < Q \leq 3537$ the 13-stage Clos interconnection.

**Conclusion.** It seems that $(2k + 1)$-stage Clos interconnections are most convenient for designing rearrangeable networks, while $3^k$-stage Karkevitch interconnections will be rather used for designing strictly non-blocking networks.

We may compute the number of crosspoints in a $(2k + 1)$-stage rearrangeable Clos interconnection for large value of $N = N'$. According to theorem 3.2 one obtains

$$k \to \ln N, \ g_2 \to 2 \ln N$$

and

$$P_{\text{opt}} \to 2N \ln N$$ (63)

when

$$N = N' \to \infty.$$ 

In the case of a $3^k$-stage strictly non-blocking Karkevitch interconnection we may write (see theorem 3.3)

$$g_3(k) = \frac{4}{3} (2\sqrt{6})^k (\frac{4}{3})^{2^k} = (\frac{4}{3})^{1 - 2^{-k}} 2^{k \log_2 (2 \sqrt{6})}.$$
When $N = N' \to \infty$ we have
\[
2^k \to \ln N \frac{\ln 2}{\ln (2 \sqrt{6})}, \quad 2^{-k} \to 0,
\]
\[
g_3(k) \to \frac{1}{3} \left[ \ln 2 \right] \log_2 (2 \sqrt{6}) \frac{(\ln N)}{(\ln (2 \sqrt{6}))} \log_2 (2 \sqrt{6}) \approx 0.20 (\ln N)^{2.29},
\]
and thus
\[
P_{\text{opt}} \to 0.20 N (\ln N)^{2.29}. \quad (64)
\]

4. Discrete optimization: a particular case

Let us recall that our main goal is to design multi-stage networks, with a minimum number of crosspoints, and performing some given function.

The results of the preceding section may be useful for that purpose. For instance, one could use theorem 3.2 in a step by step way as follows:
- choose the number of stages by rounding (50) to the nearest integer;
- choose the parameter $n_1$ by rounding (46) to the nearest integer that divide both $N$ and $N'$ (or possibly increase $N$ or $N'$);
- replace $k$ by $k - 1$, $N$ by $N/n_1$, $N'$ by $N'/n_1$ and go back to the preceding step.

However, we are not sure to obtain the best network by using this method. We may also develop methods that actually give networks with minimum number of crosspoints and are no longer based on derivation of continuous functions. A first possibility is to use the concept of finite difference of a discrete function: this has been done by Deschamps \textsuperscript{9} in the case of $(2k + 1)$-stage strictly non-blocking Clos interconnection. Ben\'es \textsuperscript{10} has presented an original method, based on a combinatorial argument, that applies to $(2k + 1)$-stage rearrangeable Clos interconnections in the case where $N = N'$. We conclude our paper by giving a generalized version of this method: it gives all the optimum rearrangeable Clos interconnections, even if $N \neq N'$.

We have to determine a nonnegative integer $k$, defining the number of stages, and $k$ positive integers $n_1, \ldots, n_k$, the parameters of the network, in order that
\[
P = \frac{NN'}{n_1 \ldots n_k} + (N + N')(n_1 + \ldots + n_k) \quad (65)
\]
be minimum (if $k = 0$, $P = NN'$) and that the product $n_1 \ldots n_k$ divides both $N$ and $N'$.

It is convenient to write (65) under an other form. Denote by $D$ the greatest common divisor of $N$ and $N'$. It is obvious that $n_1 \ldots n_k$ divides $D$. Let $n_{k+1}$
be defined as follows:

\[ n_{k+1} = \frac{D}{n_1 \ldots n_k} \]  \hspace{1cm} (66)

With that notation one obtains the following expression of \( P \):

\[ P = \frac{NN'}{D} n_{k+1} + (N + N')(n_1 + \ldots + n_k). \] \hspace{1cm} (67)

The optimization problem may be reformulated: find a factorization

\[ D = n_1 \ldots n_{k+1} \]

such that \( P \) be minimum.

**LEMMA 4.1.** If the factorization \( D = n_1 \ldots n_{k+1} \) leads to a solution of the optimization problem, then, for every \( i = 1, \ldots, k \), either \( n_i \) is a prime or \( n_i = 4 \).

**Proof.** First notice that \( n_i \) may not be equal to 1, for every \( i = 1, \ldots, k \), since the suppression of \( n_i \) in (65) would lower the value of \( P \).

If there exists a parameter \( n_i, i \in \{1, \ldots, k\} \), that is neither a prime nor 4, then

\[ n_i = n'_i n''_i, \ n'_i \geq 3 \quad \text{and} \quad n''_i \geq 2. \]

Put

\[ P' = \frac{NN'}{D} n_{k+1} + (N + N')(n_1 + \ldots + n'_i + n''_i + \ldots + n_k). \]

Let us prove that \( P' \) is smaller than \( P \). Indeed

\[ n'_i + n''_i = \frac{n_i}{n''_i} + \frac{n_i}{n'_i} \leq \frac{n_i}{2} + \frac{n_i}{3} = n_i \left( \frac{1}{2} + \frac{1}{3} \right) < n_i. \]

**Remark.** If a factorization like

\[ D = \ldots 4 \ldots n_k n_{k+1} \]

leads to a solution, then the factorization

\[ D = \ldots 2 \times 2 \times \ldots n_k n_{k+1} \]

also leads to a solution, and conversely. This is due to the fact that \( 2 + 2 = 2 \times 2 \).

**LEMMA 4.2.** If the factorization \( D = n_1 \ldots n_{k+1} \) leads to a solution of the optimization problem then

(a) \( N = N' \): \( n_{k+1} \) is either a prime or is smaller than or equal to 9;
(b) \( N \neq N \): \( n_{k+1} \) is either a prime or is smaller than or equal to 6.
Proof. If \( n_{k+1} \) is not a prime

\[
n_{k+1} = q q', \quad q \geq 2, \quad q' \geq 2,
\]

with for instance

\[
q' \geq q.
\]

Let us define

\[
P'' = \frac{N N'}{D} q' + (N + N') (n_1 + \ldots + n_k + q).
\]

One must verify that \( P \leq P'' \) that is \( P - P'' \leq 0 \):

\[
P - P'' = \frac{N N'}{D} (n_{k+1} - q') - (N + N') q = \frac{N N'}{D} n_{k+1} \left( 1 - \frac{1}{q} \right) - (N + N') q.
\]

Therefore

\[
n_{k+1} \leq \frac{N + N'}{N N'} D \frac{q^2}{q - 1}.
\]

If \( N = N' = D \) the condition (71) becomes

\[
n_{k+1} \leq \frac{2q^2}{q - 1}.
\]

If \( N \neq N' \), then

\[
D < \min (N, N')
\]

and

\[
\frac{N + N'}{N N'} D < \frac{N + N'}{\max (N, N')} < 2.
\]

Therefore in that case

\[
n_{k+1} < \frac{2q^2}{q - 1}.
\]

On the other hand one deduces from (68) that

\[
n_{k+1} \geq q^2.
\]

By combining this last inequality with (72) and (73) one obtains

\[
q^2 \leq \frac{2q^2}{q - 1} \quad \text{and} \quad q^2 < \frac{2q^2}{q - 1},
\]
that is \( q \leq 3 \) and \( q < 3 \), respectively.

If \( N = N' \), the value of \( q \) may be 2 or 3; therefore

\[
q = 2 \quad n_{k+1} \leq 8, \quad q = 3 \quad n_{k+1} \leq 9.
\]

If \( N \neq N' \), the value of \( q \) is 2; therefore

\[
n_{k+1} < 8
\]

that is, since \( n_{k+1} \) is not a prime,

\[
n_{k+1} \leq 6.
\]

LEMMA 4.3. If the factorization \( D = n_1 \ldots n_{k+1} \) leads to a solution of the optimization problem then

\[
\begin{align*}
(a) \quad & \frac{NN'}{D} < N + N' \quad n_{k+1} \geq n_i \quad (\forall i = 1, \ldots, k). \\
(b) \quad & \frac{NN'}{D} = N + N' \quad n_{k+1} \text{ is a prime or is equal to 4.} \\
(c) \quad & \frac{NN'}{D} > N + N' \quad n_{k+1} \leq n_i \quad (\forall i = 1, \ldots, k).
\end{align*}
\]

Proof. (a) and (c). In the contrary case, the permutation of \( n_{k+1} \) and \( n_i \) yields a better solution. (b) Same proof as lemma 4.1.

Remark. Let us suppose that

\[
D = p_1^{a_1} \ldots p_q^{a_q}, \quad p_1 < \ldots < p_q, \quad \alpha_i > 0 \quad \text{and} \quad p_i \text{ prime,}
\]

for every \( i = 1, \ldots, q \).

In case (a) one must have \( n_{k+1} \geq p_q \) while in case (c) one must have either \( n_{k+1} = p_1 \) or \( n_{k+1} = 4 \).

We are now in place to examine the various cases than can arise.

Discussion

(1) \( N = N' \left( \text{and thus} \frac{NN'}{D} = N < N + N' \right) \).

(1.1) \( N = p_1^{a_1} \ldots p_q^{a_q} \) with \( p_q \geq 11 \);

\[
n_{k+1} = p_q
\]
(1.2) \( N = 2^w 3^x 5^y 7^z \) with \( z \geq 1 \).

\[
\begin{align*}
  n_{k+1} &= 7 \quad P = 7N + 2N(2w + 3x + 5y + 7z) - 14N; \\
  n_{k+1} &= 8 \quad P = 8N + 2N(2w + 3x + 5y + 7z) - 12N; \\
  n_{k+1} &= 9 \quad P = 9N + 2N(2w + 3x + 5y + 7z) - 12N.
\end{align*}
\]

The smallest \( P \) is obtained when \( n_{k+1} = 7 \).

(1.3) \( N = 2^w 3^x 5^y \) with \( y \geq 1 \).

A similar reasoning shows that \( n_{k+1} = 5 \).

(1.4) \( N = 2^w 3^x \) with \( x \geq 1 \).

\[
\begin{align*}
  n_{k+1} &= 3 \quad P = 3N + 2N(2w + 3x) - 6N; \\
  n_{k+1} &= 4 \quad P = 4N + 2N(2w + 3x) - 8N; \\
  n_{k+1} &= 6 \quad P = 6N + 2N(2w + 3x) - 10N; \\
  n_{k+1} &= 8 \quad P = 8N + 2N(2w + 3x) - 12N; \\
  n_{k+1} &= 9 \quad P = 9N + 2N(2w + 3x) - 12N.
\end{align*}
\]

For \( n_{k+1} = 4, 6 \) or 8 one obtains \( P = 2N(2w + 3x) - 4N \); for \( n_{k+1} = 3 \) or 9 one obtains \( P = 2N(2w + 3x) - 3N \).

Therefore, one chooses \( n_{k+1} \) among 4, 6 or 8 if one of these numbers divides \( N \); in the contrary case one chooses \( n_{k+1} \) among 3 and 9. This is summarized by the following graph

\[
\begin{array}{c|c}
3, 9 & \\ \\
| & \downarrow \\
4, 6, 8 & \\ \\
\end{array}
\]

in which the lower written values for \( n_{k+1} \) give the lower values for \( P \).

(1.5) \( N = 2^w \) with \( w \geq 1 \).

One obtains \( n_{k+1} = 4 \) or 8 if \( w \geq 3 \). If \( w \leq 2 \), one can easily prove that the solution is \( k = 0 \) and \( P = NN' \).

(2) \( N \neq N' \) and \( \frac{NN'}{D} < N + N' \).

(2.1) \( D = p_1^{a_1} \ldots p_q^{a_q} \) with \( p_q \geq 7 \);

\[
  n_{k+1} = p_q.
\]

(2.2) \( D = 2^w 3^x 5^z \), with \( z \geq 1 \).

\[
\begin{align*}
  n_{k+1} &= 5 \quad P = 5\frac{NN'}{D} + (N + N')(2w + 3y + 5z) - 5(N + N'); \\
  n_{k+1} &= 6 \quad P = 6\frac{NN'}{D} + (N + N')(2w + 3y + 5z) - 5(N + N'); \\
  n_{k+1} &= 5.
\end{align*}
\]


(2.3) \( D = 2^w 3^x \), with \( x \geq 1 \).

\[
\begin{align*}
n_{k+1} &= 3 \quad P = 3 \frac{NN'}{D} + (N + N')(2w + 3x) - 3(N + N'); \\
n_{k+1} &= 4 \quad P = 4 \frac{NN'}{D} + (N + N')(2w + 3x) - 4(N + N'); \\
n_{k+1} &= 6 \quad P = 6 \frac{NN'}{D} + (N + N')(2w + 3x) - 5(N + N').
\end{align*}
\]

Let us denote by \( P(i) \) the value of \( P \) when \( n_{k+1} = i \).

\[
P(3) - P(4) = (N + N') - \frac{NN'}{D} > 0;
\]

\[
P(6) - P(4) = 2 \frac{NN'}{D} - (N + N') > 0 \text{ since } \frac{N + N'}{NN'} D < 2
\]
as it has been seen in the proof of lemma 4.2;

\[
P(3) - P(6) = 2(N + N') - 3 \frac{NN'}{D}.
\]

One obtains the three following graphs, according to the relative magnitude of \( \frac{NN'}{D} \) and \( \frac{3}{2}(N + N') \):

(a) \( \frac{NN'}{D} < \frac{3}{2}(N + N') \)

\[
\begin{array}{c|c|c}
3 & \frac{6}{4} \\
6 & 4
\end{array}
\]

(b) \( \frac{NN'}{D} = \frac{3}{2}(N + N') \)

\[
\begin{array}{c|c|c}
3, 6 & \frac{1}{4} \\
6 & 4
\end{array}
\]

(c) \( \frac{NN'}{D} > \frac{3}{2}(N + N') \)

\[
\begin{array}{c|c|c}
6 & \frac{3}{4} \\
3 & 4
\end{array}
\]

(2.4) \( D = 2^w \) with \( w \geq 1 \). One obtains \( n_{k+1} = 4 \) if \( w \geq 2 \) and \( n_{k+1} = 2 \) if \( w = 1 \).

(2.5) \( D = 1, k = 0 \) and \( P = NN' \).

(3) \( \frac{NN'}{D} > N + N' \).
(3.1) \( D = p_1^{a_1} \ldots p_g^{a_g} \) with \( p_1 \geq 3; \)
\[ n_{k+1} = p_1. \]

(3.2) \( D = 2^w 3^x \ldots \) with \( w \geq 1. \)
\[ n_{k+1} = 1 \quad P = \frac{NN'}{D} + (N + N')(2w + 3x + \ldots); \]
\[ n_{k+1} = 2 \quad P = 2\frac{NN'}{D} + (N + N')(2w + 3x + \ldots) - 2(N + N'); \]
\[ n_{k+1} = 4 \quad P = 4\frac{NN'}{D} + (N + N')(2w + 3x + \ldots) - 4(N + N'). \]
\[
P(4) - P(2) = 2\frac{NN'}{D} - 2(N + N') > 0; \]
\[
P(1) - P(2) = 2(N + N') - \frac{NN'}{D}. \]

(a) \( \frac{NN'}{D} < 2(N + N') \quad n_{k+1} = 2; \)

(b) \( \frac{NN'}{D} = 2(N + N') \quad n_{k+1} = 1 \) or \( 2; \)

(c) \( \frac{NN'}{D} > 2(N + N') \quad n_{k+1} = 1. \)

(3.3) \( D = 1. \quad P = NN'. \)

Remark. The central stage contains commutators with \( Nn_{k+1}/D \) inputs and \( N'n_{k+1}/D \) outputs. Therefore the situation where \( n_{k+1} = 1 \) is not unrealistic: it only occurs when
\[
\frac{NN'}{D} > 2(N + N')
\]
and this implies
\[
\frac{\min (N, N')}{D} > \frac{2(N + N')}{\max (N, N')} > 2,
\]
\[
\frac{\max (N, N')}{D} > \frac{2(N + N')}{\min (N, N')} > 4.
\]
In conclusion, all the solutions are obtained as follows:

(1) Choose \( n_{k+1} \) according to the above discussion.

(2) For every \( n_{k+1} \), factorize \( \frac{D}{n_{k+1}} \) in prime factors:

\[
\frac{D}{n_{k+1}} = p_1^{a_1} p_2^{a_2} \ldots p_q^{a_q};
\]

choose the parameters \( n_1, \ldots, n_k \) according to the following rules:

(2.1) If \( p_1 = 2 \) and \( \alpha_1 = 1 \), or if \( p_1 \geq 3 \), then there are \( \alpha_1 \) \( p_1 \)'s, \( \alpha_2 \) \( p_2 \)'s, \ldots, \( \alpha_q \) \( p_q \)'s among the \( n_i \)'s, \( i = 1, \ldots, k = \alpha_1 + \alpha_2 + \ldots + \alpha_q \).

(2.2) If \( p_1 = 2 \) and \( \alpha_1 \geq 2 \) there are several choices: one must have \( \alpha'_1 \) \( 2 \)'s, \( \alpha''_1 \) \( 4 \)'s, \( \alpha_2 \) \( p_2 \)'s, \ldots, \( \alpha_q \) \( p_q \)'s among the \( n_i \)'s, with the condition that

\[
\alpha'_1 + 2\alpha''_1 = \alpha_1.
\]

Notice that all along the discussion we have generally not taken into account the possibility that the solution be obtained for \( k = 0 \). Therefore, in order to be complete, one should verify that the obtained value of \( P \) is smaller than \( NN' \).

**Examples**

(1) \( N = N' = 2^{15} \).

(a) \( n_{k+1} = 4 \); it remains to factorize \( 2^{13} \).

Typical solutions: \( n_1 = \ldots = n_{13} = 2; \)
\[
\begin{align*}
n_1 = \ldots = n_9 = 2, & \quad n_{10} = n_{11} = 4; \\
n_1 = \ldots = n_7 = 2, & \quad n_8 = n_9 = n_{10} = 4; \\
n_1 = \ldots = n_5 = 2, & \quad n_6 = \ldots = n_9 = 4; \\
n_1 = n_2 = n_3 = 2, & \quad n_4 = \ldots = n_8 = 4; \\
n_1 = 2, & \quad n_2 = \ldots = n_7 = 4.
\end{align*}
\]

(b) \( n_{k+1} = 8 \); it remains to factorize \( 2^{12} \).

Typical solutions: \( n_1 = \ldots = n_{12} = 2; \)
\[
\begin{align*}
n_1 = \ldots = n_{10} = 2, & \quad n_{11} = 4; \\
n_1 = \ldots = n_8 = 2, & \quad n_9 = n_{10} = 4; \\
n_1 = \ldots = n_6 = 2, & \quad n_7 = n_8 = n_9 = 4; \\
n_1 = \ldots = n_4 = 2, & \quad n_5 = \ldots = n_8 = 4; \\
n_1 = n_2 = 2, & \quad n_3 = \ldots = n_7 = 4; \\
n_1 = \ldots = n_6 = 4.
\end{align*}
\]

(2) \( N = 2^{16}, \quad N' = 2^{13} \).

\[
D = 2^{13}, \quad \frac{NN'}{D} = N < N + N';
\]

\( n_{k+1} = 4 \); it remains to factorize \( 2^{11} \).
Typical solutions:

\[ n_1 = \ldots = n_{11} = 2; \]
\[ n_1 = \ldots = n_9 = 2, n_{10} = 4; \]
\[ n_1 = \ldots = n_7 = 2, n_8 = n_9 = 4; \]
\[ n_1 = \ldots = n_5 = 2, n_6 = n_7 = n_8 = 4; \]
\[ n_1 = n_2 = n_3 = 2, n_4 = \ldots = n_7 = 4; \]
\[ n_1 = 2, n_2 = \ldots = n_6 = 4. \]

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