In October 1959, Professor Dr. Balthasar van der Pol died at Wassenaar at the age of 70. In the years from 1922 to 1949, during which he acquired international repute in the field of radio science and engineering, he directed the work done in that field in Philips Research Laboratory at Eindhoven. In the same period he also played a prominent part in international organizational activities concerned with radio communications. After his retirement from industry he continued in these activities, principally as Director of the C.C.I.R. (Comité Consultatif International des Radiocommunications) at Geneva, an office he held until a few years before his death. Amongst the numerous distinctions conferred upon him were the Medal of Honor of the Institute of Radio Engineers, of which he was for some time vice-president, the Valdemar Poulsen Gold Medal, several honorary doctorates, and the honorary presidency of the U.R.S.I. (Union Radio Scientifique Internationale).

The obituary notices that appeared in journals both in the Netherlands and abroad were obviously unable to go far into the details of his scientific work. It gives the editors pleasure to be able to publish here a more comprehensive appreciation, written by Prof. Bremmer, one of Van der Pol's former and closest collaborators and probably the most familiar with his work. In our view such an appreciation is a fitting way of paying tribute to the memory of this remarkable man, and at the same time it gives the reader a glimpse of many important aspects of the evolution of radio science.

The title photograph shows Van der Pol (right) in conversation with Prof. Holst, the founder and first director of this laboratory.

THE SCIENTIFIC WORK OF BALTHASAR VAN DER POL

by H. BREMMER.
the theoretical-mathematical treatment of technical problems, and also to meet professional mathematicians on their own ground and persuade them of the importance of a rigorous mathematical approach to such problems. He saw how difficult it often is to bring together the practitioners in these three fields. His success in this respect was a valuable facet of his life’s work.

As a mathematician, Van der Pol showed a strong preference for heuristic methods, by which, along partly intuitive lines, results are readily arrived at that can only be verified later. To this extent he resembled Heaviside. This brilliant man, who lived from 1850 to 1925, used mathematical methods which proved to be very fruitful, but generally left it to others to demonstrate their validity. Heaviside was nevertheless the first to get to the bottom of such technical problems as the role of inductances in long-distance telephone cables. Van der Pol was a great admirer of this self-made scientific genius, as testified by his inaugural address on 8th December 1938 as extra-mural professor at Delft 1). The address was entirely devoted to Heaviside, and underlined the significance of his work at a time when the application of Maxwell’s equations was not yet a commonplace. The following sentence from Van der Pol’s address will serve to illustrate the part played by Heaviside in the development of electrotechnical concepts that are now thoroughly familiar:

“Electrotechnology and physics are also indebted to Heaviside for the concept and the word “impedance”, not merely as understood in present-day alternating-current theory, but in a very much wider sense which includes on-off switching effects, a form which even today has still not been dealt with exhaustively.”

In all his work Van der Pol owed much to the inspiration of Heaviside. This appears directly from his preoccupation with the methods of operational calculus, which were created by Heaviside for application to electrical networks. No one has explained better than Van der Pol the merits of operator methods. In particular he demonstrated their value for the discovery of numerous relations in widely diverse fields of mathematics. In one respect, however, there was a marked contrast between Heaviside and Van der Pol. The former suffered all his life from the difficulty of getting his ideas accepted, not least because of the almost negligible form of his writings. The latter’s publications, on the other hand, were models of careful exposition and clarity of thought.

Van der Pol’s lucidity was also much in evidence in the lectures he gave and in the innumerable meetings and discussions over which he presided. In this connection, mention must be made of his considerable talents as an organizer. The ease with which he could present a matter, and his complete familiarity with the whole field of radio, soon made him a leading figure in international organizations, especially in the U.R.S.I. and the C.C.I.R. It was no surprise when, after retiring from Philips Research Laboratory, he was invited to be the first director of the C.C.I.R. upon the establishment of its permanent secretariat. In the time that he held this office (from 1949 to 1956) he worked with devotion to ensure that justice was done to pure scientific research in the advice issued on the preparation of technical regulations for international radio communications. It is scarcely believable that, with all his activities on international committees, he should still have found the time for abstract investigations. He succeeded in this only because his zest for work was quite out of the ordinary, and remained with him to the end of his life.

Van der Pol maintained that only a few great men have had a decisive bearing on the development of physics. He used to say that, faced with the enormous number of scientific investigations, it was easy to forget that the really fundamental work is contained in a mere handful of publications. He himself was therefore strongly drawn towards personal contact with the leading figures in the world of science. In that respect he was fortunate at the very beginning of his career. After taking his degree cum laude at Utrecht in 1916, he did experimental research in England until 1919. At first he worked in London under Fleming, the man who, in 1904, was granted the first patent on a diode. There followed a period of two years at Cambridge, where he worked under Sir J. J. Thomson. On his return to the Netherlands he was attached for three years to the Teyler Foundation at Haarlem, where he worked under Lorentz. Thus, Van der Pol was in close association with two scientists, one of whom he himself wittily described as the “discoverer” of the electron (in 1897 Thomson had determined the ratio e/m from electron orbits in a combined electrical and magnetic field) and the other as the “inventor” of the electron (in 1896 Lorentz had calculated e/m from the Zeeman effect). In the years that followed, Van der Pol was in close contact with other distinguished scientists, including Appleton, who was awarded the Nobel prize in 1947 for his pioneering research on the ionosphere.

Propagation of radio waves

In England Van der Pol soon came up against a problem which was to remain one of his great lifelong interests. We refer to the effect of the earth on the propagation of radio waves. In 1909 Sommerfeld \(^2\) had put forward a theory which explained the observed phenomena on the assumption that the earth's surface could be regarded as flat. As early as 1901, however, Marconi had succeeded in sending radio signals across the Atlantic. In 1915, after experience had been gained in the use of undamped waves, it had even proved possible to establish telephonic communication by radio across the Atlantic. It was evident that the curvature of the earth must have some effect on the propagation of the waves, because linear propagation straight through the highly absorbent earth would attenuate the signal beyond possibility of detection. It was conceivable that, as a result of diffraction, the waves might follow the surface of the earth far beyond the horizon as optically observed from the transmitting aerial. The existing theory on this idea was criticized by Poincaré, who showed in 1910 \(^3\) that, in the case of diffraction, the field over great distances must decrease proportionally to \(\exp \left(-\beta D/\lambda^2\right)\), where \(\lambda\) is the wavelength and \(D\) the distance from the transmitter to the receiver measured over the earth's surface. Unfortunately, it was not at that time possible to test this theory, since there was no way of deriving a reliable numerical value for the constant \(\beta\).

This was roughly the situation when Van der Pol first came into touch with the propagation problem. Unless the value of \(\beta\) was extremely small, the field upon diffraction would be so strongly attenuated that there could only be one other explanation for the long range of the radio waves: the presence of the ionosphere. The existence of conducting layers at high altitudes in the atmosphere, known as the ionosphere, was postulated in 1902 by Kennelly and Heaviside to account for the propagation of waves over great distances. The radio waves would then follow a zig-zag path, being reflected successively from the ionosphere and from the surface of the earth.

The mathematical difficulties of the pure diffraction problem, assuming the absence of the ionosphere, were bound up with the numerous Bessel functions occurring in the solution. This induced Van der Pol to encourage the mathematician Watson to study this problem, at a time when the latter was working on his well-known book on Bessel functions, which appeared in 1922. Watson's results, which were published in 1918 \(^4\), proved beyond doubt that the value of \(\beta\) was too large (it would be 0.00376 km\(^{-1}\) if the earth were a perfect conductor and the atmosphere completely homogeneous) to explain the long range of radio waves without invoking the help of the ionosphere. The implications of Watson's calculations were discussed by Van der Pol in a paper published in 1919 \(^5\).

This did not mean, however, that further study of the pure diffraction problem, leaving the ionosphere out of account, was no longer necessary. In the first place the wave propagated along the earth, the "ground wave", makes an important contribution over short distances, particularly during the day, compared with the contribution from the "sky wave" which is propagated via the ionosphere. Moreover, the ionosphere has little effect on the extremely short waves used in television, frequency-modulated transmissions and radar. The further study of the ground wave was therefore still of importance. In cooperation with K. F. Niessen, Van der Pol elaborated on Sommerfeld's theory \(^6\), which could still be used for short distances, and made a special study of the influence of the height of the transmitting and receiving aerials above the ground. In later years the ground-wave theory was worked out by Van der Pol and the present author for distances at which the earth's curvature plays an essential part \(^7\). A mathematical method was developed for computing the fields of the ground wave for all topographical conditions. It was found that, even in the case of very short waves, the decrease of the field strength beyond the transmitter horizon was very much more gradual than had formerly been thought. This is illustrated in fig. 1, where the ratio of the actual field strength \(E\) on the earth's surface to the field strength \(E_{pr}\) in free space is plotted as a function of the distance from transmitter to receiver for a given case, viz. a transmitter at a height of 100 metres above the earth's surface and surface conditions corresponding to dry ground. The various curves show this ratio for four different wavelengths (differing successively from the other by a factor of 10) and for the limiting case \(\lambda \rightarrow 0\). A distinct shadow effect, whereby the field strength is reduced by a factor

---

\(^5\) B. van der Pol, Phil. Mag. 38, 365, 1919.
\(^7\) B. van der Pol and H. Bremmer, Phil. Mag. 24, 141 and 825, 1937.
of at least 4 within a distance of 5 km beyond the horizon, does not appear until the wavelength has dropped to about 7 mm.

The numerical results of the ground-wave propagation theory are of particular importance at wavelengths smaller than about 10 m, in which case the heights of the transmitting and receiving aerials above the earth's surface have a considerable influence. During Van der Pol's term of office with the C.C.I.R. a wide programme of computations was decided upon in order to get numerical data on ground-wave propagation, including the effect of refraction in the lowest layer of the atmosphere. The results were published in the form of graphs 8), an example of which is shown in fig. 2; this refers to a wavelength of 5 m, again for representative ground conditions (conductivity of 0.01 mho/m and a relative dielectric constant of 10). The graph indicates, for example, the extent to which the field increases with increasing height \( h_2 \) of the receiver for a fixed transmitter height \( h_1 \) of 10 m. The results are the same if the transmitter and receiver heights are interchanged.

The rigorous theory on which such data are based is applicable to all problems of waves meeting a spherical obstacle. This situation also arises when light rays refracted in spherical drops of water give rise to rainbow effects. These considerations accordingly led to a new treatment of the rainbow in terms of wave theory 7)*.

We shall now return to those cases in which the sky wave conducted via the ionosphere is more important than the ground wave. As shown by Watson's calculations, mentioned above, this is the normal state in short-wave transmissions over great distances. The important role played by the ionosphere, established beyond all doubt by Watson's results, was later emphasized by Van der Pol in the following sentence from his above-mentioned inaugural speech:

"You know of that conductive layer, high in the atmosphere, which makes radio communication over long distances possible, and whose existence is just as important to radio technology as the existence of iron in the earth is to electrical engineering as a whole".


*) Editorial note: By coincidence, a part of this wave treatment of the rainbow is to be seen scribbled on the blackboard in the title photograph.
Although the existence of the ionosphere was not definitely established until 1925, when direct reflection measurements placed it beyond all doubt, Eccles and Larmor had worked out a theory as early as 1912 for wave propagation through a conductive gas, which could be immediately applied to the ionosphere. It was found that a conductive gas behaves formally like a medium possessing a time an apparent dielectric constant smaller than unity. This represented experimental confirmation of the now generally accepted propagation mechanism for radio waves that are refracted in the ionosphere. This whole subject was clearly dealt with in Van der Pol's thesis in 1920, entitled "The influence of an ionized gas on the propagation of electromagnetic waves, and its applications in the
certain conductivity and a relative dielectric constant smaller than unity. This means that the phase velocity of radio waves in the ionosphere exceeds the speed of light $c$ in a vacuum. However, the corresponding propagation velocity of discontinuities (group velocity) is smaller than $c$, as it must be to accord with the principles of the theory of relativity. By means of resonance measurements on lecher lines, Van der Pol determined, during his period in the Cavendish Laboratory at Cambridge, the conductivity and dielectric constant of the plasma in glow discharges; these experiments closely simulated the conditions in the ionosphere. After experimental difficulties had been overcome, it was possible in this way to measure for the first field of wireless telegraphy and in measurements on glow discharges”.

Non-linear circuits: relaxation oscillations

We shall now consider another field of research in which Van der Pol was particularly active, that of non-linear oscillations. It was understandable that in the 'twenties his attention should be drawn to this subject, since the advances already made in the application of triode circuits made necessary a deeper understanding of oscillation phenomena in order to round off the theory and to discern further possibilities. The simple linear theory of oscillatory effects is arrived at when the $i_a v_g$ characteristic

---

**Fig. 2.** Vertical component of the field strength of the ground wave (in dB and in $\mu V/m$) of a transmitter at height $h_1 = 10$ m, computed for various distances $D$ and receiver heights $h_2$, for $\lambda = 5$ m and over slightly moist ground. The dashed curve would be obtained in the absence of the earth. The point on each curve indicates the horizon distance when the line connecting the transmitter and receiver touches the earth.
of the triode (anode current versus grid voltage) is approximated by a straight line. The unsatisfactory point is that the amplitude of the excited oscillations then remains undetermined. The limitation of the amplitude is due to the decrease of the slope of the characteristic at either side of the operating point.

Fig. 3 shows a representative circuit for investigating these effects, in which $M$ is the mutual inductance between the grid circuit and the $LC$ circuit (with shunt resistance $R$) in the anode lead. We take the triode characteristic to be given by:

$$i_a = \Phi(v_a + g v_g), \ldots (1)$$

where $g$ is the amplification factor of the triode. The non-linear function $\Phi$ of the variable $u = v_a + g v_g$ can be represented by a Taylor series in the neighbourhood of the point $u = E_a$:

$$\Phi(u) = \Phi(E_a) + a \left( \frac{u - E_a}{k} \right) +$$

$$+ \beta \left( \frac{u - E_a}{k} \right)^2 - \gamma \left( \frac{u - E_a}{k} \right)^3 + \ldots \ldots (2)$$

The coefficients are normalized by the introduction of the parameter $k = g(M/L) - 1$. This circuit leads to the following differential equation for the time variation of the voltage $v$ (see figure):

$$\frac{d^2v}{dt^2} + \left\{ \frac{1}{R} - a \right\} + 2\beta v + 3\gamma v^2 + \ldots \left\{ \frac{dv}{dt} + \frac{1}{L} v = 0. \ldots (3)$$

Van der Pol showed in the first place that the coefficient $\beta$, which determines the first non-linear term of the characteristic, so important for detection, does not in itself establish a finite value of the amplitude. The effect of this term will merely be to cause the angular frequency of the $LC$ circuit to differ slightly from its value $(LC)^{-1}$ in the linear theory. The term with the coefficient $\gamma$ is the first term to influence the amplitude. The following terms determine less essential properties. For example the fifth-order term in (2) is necessary in order to understand certain hysteresis effects, also studied by Van der Pol\(^9\). In his amplitude investigations he therefore confined himself mainly to study the influence of the $\gamma$ term, taking $\beta = 0$. By neglecting all higher terms in the expansion of the characteristic, eq. (2), he obtained from (3) a differential equation which, after introducing reduced variables both for the voltage $v$ and the time $t$, assumed the form:

$$\frac{d^2y}{dx^2} - \varepsilon(1 - y^2) \frac{dy}{dx} + y = 0, \ldots \ldots (4)$$

where $\varepsilon = \sqrt{L/C} (a - 1/R)$.

This differential equation is known as "Van der Pol's equation". Initial investigation of this equation for small values of the dimensionless parameter $\varepsilon$ showed that the solution, after a preliminary "time" $x_0$ of the order of $1/\varepsilon$, approximates to a normal sinusoidal oscillation of amplitude equal to 2. Van der Pol wondered what the corresponding solution would look like for larger values of $\varepsilon$, at least of the order of unity. The mathematical difficulties involved led him to apply a graphical procedure, the method of isoclines\(^10\). In the case under consideration this amounts to reducing equation (4) with the aid of the new variable $dy/dx = z$ to the following first-order differential equation:

$$\frac{dz}{dy} = \varepsilon(1 - y^2) - \frac{y}{z}. \ldots \ldots (5)$$

Plotting $z$ against $y$, this equation determines at every point a slope $dz/dy$, which can be indicated by a short dash. Drawing a continuous curve through a series of these dashes produces a curve which represents a possible solution in the variables $y$ and $z$. A start can be made at various points, and in this way solutions constructed that satisfy the relevant initial conditions. Further numerical integration produces from each curve a solution for $y$ as a function of $x$.

This procedure is illustrated in fig. 4, where various solutions are given for the relation between $y$ and $dy/dx$, when $\varepsilon = 1$. Fig. 5 shows the solution for $y$ itself, corresponding to one of these, and also solutions for $y$ for the cases $\varepsilon = 0.1$ and $\varepsilon = 10$, similarly.

---

\(^9\) B. van der Pol, Phil. Mag. 43, 700, 1922.

\(^10\) B. van der Pol, Phil. Mag. 2, 978, 1926.
Fig. 4. Illustrating the isocline method of solving the differential equation (5), for $\varepsilon = 1$.

obtained. It follows from fig. 5 that, as the time increases, the solution approaches asymptotically to a non-sinusoidal periodic function, which differs more from a sinusoidal oscillation the larger $\varepsilon$ is; the larger the value of $\varepsilon$ the sooner is the asymptotic end state reached. Furthermore, the period in this end state for large values of $\varepsilon$ is seen to approach a value $\Delta x$ which is of the order of $\varepsilon$; it thus differs appreciably from the corresponding period for very small values of $\varepsilon$, which, in terms of the reduced time unit $x$, approaches $2\pi$. Returning to the original variables $v$ and $t$ in eq. (3), this means that the period for large values of $\varepsilon$ is of the order of magnitude of $L(a - 1/R)$. This is a relaxation time, i.e. a time that determines the decay of an aperiodic process. (Better known is the case occurring in other circuits where a relaxation time is determined by an $RC$ product, see e.g. fig. 6.) For this reason, Van der Pol gave the name relaxation oscillations to these oscillatory effects which, at large values of $\varepsilon$, constitute the end state of processes defined by the differential equation (4).

Van der Pol first described these oscillations in 1926, after which he continued to investigate them, theoretically and experimentally, in cooperation with J. van der Mark [11]. A particular study was made of forced oscillations [12], produced when an external alternating voltage is applied to a system like that in fig. 3. In the simplest cases the situation can be described by substituting for the right-hand side of Van der Pol's equation a term proportional to $\cos \omega t$. For small values of $\varepsilon$ this provided a better understanding of already known properties of oscillator circuits. This applied particularly to the suppression of the free oscillation of a system that occurs when its frequency is close to that of the imposed external oscillation. The resultant synchronism with the latter oscillation was found to be particularly pronounced when, with increasing $\varepsilon$, the first, approximately sinusoidal, free oscillation gradually changes into a free relaxation oscillation. Another way of putting this is that the relaxation oscillations very readily assume the frequency of a superposed alternating voltage if the latter does not

11) B. van der Pol and J. van der Mark, Onde électrique 6, 461, 1927.
not differ too much from the frequency, determined by a relaxation time, of the natural free relaxation oscillations. If the circuit be modified so as to reduce this relaxation time to about half its original value, fairly abrupt synchronization will occur with half the frequency of the external oscillation, and after a further change with a third of this frequency, and so on. In other words, it is possible to generate relaxation oscillations that can successively be synchronized with an external oscillation and its sub-harmonics. These synchronization effects occur the more readily the larger the value of $s$, or more generally, the more numerically important are the higher terms in eq. (2). Whilst the frequency of a relaxation oscillation can thus very easily be influenced, the amplitude is not nearly so readily controlled.

This synchronization with sub-harmonics later proved to be of great practical importance in the early development of television \textsuperscript{13).} A relaxation oscillation is used in television for the scanning of successive lines of the picture. With suitable circuitry a succession of sub-harmonics can be formed, finally producing a relaxation oscillation with a synchronized frequency of 25 (or 30) c/s. This is then used to effect the return to the first line after the complete scanning of each frame. A simple circuit for demonstrating the frequency division of relaxation oscillations is shown in fig. 6, where frequency of the free relaxation oscillations (which is proportional to $1/(RC)$) by gradually increasing the capacitance $C$, it is easy to demonstrate the sudden occurrence of synchronization with the successive subharmonics having frequencies $f_0/2$, $f_0/3$, $f_0/4$ and so on. This is further illustrated in fig. 7, where

![Fig. 6. Circuit for demonstrating the frequency division produced by the synchronization of relaxation oscillations with the sub-harmonics of an imposed voltage $E_0 \sin 2\pi f_0 t$.](image)

![Fig. 7. Synchronization with sub-harmonics. The graph represents the period $1/f$ of the forced relaxation oscillation of the circuit in fig. 6 excited at a constant frequency $f_0$ while the capacitance $C$ is gradually varied.](image)

The variable capacitance $C$ is set out along the abscissa. The stepwise changes in the period of the forced oscillation can be read from the ordinate.

Subsequently, in collaboration with C. C. J. Addink, Van der Pol entered upon the study of more general synchronization phenomena. It proved possible to obtain synchronization not only with frequencies $f_0/2$, $f_0/3$, $f_0/4$ etc., but also with frequencies $(n/m)f_0$, where $n$ and $m$ are arbitrary integers ($n$ may even be $> m$). For these experiments the frequency of the imposed oscillation was varied instead of one of the parameters of the relaxation oscillator \textsuperscript{14).}

Characteristic of the operation of the circuit in fig. 6 is the occurrence of an essentially aperiodic process (in this case the exponential charge and discharge of the capacitor) which is periodically interrupted and restarted at certain critical values of a relevant parameter. Such situations are frequently found in nature, and they always give rise to relaxation oscillations. Van der Pol was struck by their very number and variety. He drew up a long list of examples, which included: the aeolian harp, the scratching of a knife on a plate, the periodic recurrence of epidemics and economic crises, the periodic density variation of two (or another even number) types of animals that live together

\textsuperscript{13) J. van der Mark, An experimental television transmitter and receiver, Philips tech. Rev. 1, 16-21, 1936.}

\textsuperscript{14) B. van der Pol, Actualités scientifiques et industrielles No. 718, published by Hermann, Paris 1938, p. 69-80.}
and of which one serves as food for the other, the sleep of flowers, phenomena associated with periodic rain showers after the passing of a meteorological depression, and finally the beat of the heart.

The latter example led Van der Pol to a very interesting and instructive practical application of relaxation oscillations, namely an electrical model for simulating all the rhythmic movements of the human heart. With this system it was a simple matter to record “electrocardiograms”, and to study the normal heart beat as well as cardiac disorders. The importance of the system to medical science will be obvious. The heart may be regarded as consisting of three systems: the sinus venosus, the two auricles and the two ventricles. The operation of each system can be simulated by a circuit of the type in fig. 6. The unilateral effect of the sinus on the auricles, and of the auricles on the ventricles, was represented by non-amplifying triodes. After other details of the operation of the heart had been taken into account the model illustrated in figs. 8 and 9 was finally produced. The ignition of the gas discharges in the circuits simulating the auricles and ventricles represents respectively an auricular and a ventricular systole. The synchronizations in this model with subharmonics of the resultant dominant heart beat can be recognized in the human heart in certain cases.

The development of the heart model out of the work on relaxation oscillators was a side-track similar to that which led from considerations of radio-wave propagation to a new wave-theoretical treatment of the rainbow. Van der Pol liked to point out how radio problems were apt to draw attention to fields which, at first sight, seem to have very little connection with radio.

### Transient phenomena and operational calculus

Operational calculus was introduced by Heaviside with the original object of providing a simple means

---

15) B. van der Pol and J. van der Mark, Phil. Mag. 6, 763, 1928.
16) The examples mentioned and various others will be found in: B. van der Pol, Proc. World Radio Convention, Sydney 1938.
of studying switching transients in electrical networks. The methods devised were later developed into a system of calculus which has proved to be extremely useful in mathematics as a whole. No one saw this more clearly than Van der Pol, who demonstrated by numerous examples the possible applications of operational methods to widely diverse fields of study.

To illustrate the starting point of operational calculus we shall briefly discuss an aspect of the historical development of alternating current circuit theory, often examined by Van der Pol. At first the aim was to determine how a certain system responds to an externally applied harmonic vibration \( \cos \omega t \), or, more generally, \( \exp j \omega t \), where \( \cos \omega t \) is the real component. Investigations were made to ascertain in what way the properties of the system depend on the frequency \( f = \omega/2\pi \). As a case in point we shall consider an amplifier. The ratio \( G(j\omega) \) between the output voltage and input voltage were studied in the case where both were proportional to \( \exp j \omega t \). If the function \( G(j\omega) \) is known, we can calculate with the aid of Fourier analysis the output voltage for any arbitrary time function of the input voltage. The case of the response to a discontinuous input voltage can also be treated in this way. In applying this procedure it was often overlooked, however, that the complex function \( G(j\omega) \) depends on two real (though mutually related) functions, and that it is consequently not enough to take account only, for example, of the modulus \( |G(j\omega)| \). This was discussed some considerable time ago in this journal by J. Haantjes 17).

On the other hand, all properties can equally be derived from the knowledge of a single real function, such as the unit step function \( G_s(t) \). This is defined as the output voltage resulting from the application at a given moment \( t = 0 \) of a direct voltage of unit magnitude to the input side, when no voltage was present for \( t < 0 \). (By giving the input function unit voltage, \( G_s \) is a dimensionless function of time.) All characteristics of the amplifier can be described in terms of this unit function. It can be shown that for a given arbitrary primary voltage \( v_{pr}(t) \) at the input, the output voltage of the amplifier will be

\[
v(t) = \int G_s(t - \tau) \frac{dv_{pr}(\tau)}{d\tau} d\tau = \int G_s(\tau) \frac{dv_{pr}(t-\tau)}{d\tau} d\tau.
\]  

This representation is simpler than that where the system is described in terms of the above-mentioned complex function \( G(j\omega) \), which defines the behaviour of the system in the case of a continuous, harmonic input voltage. If the latter function is used, the effect of an arbitrary input voltage \( v_{pr}(t) \) is given by the expression:

\[
v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(j\omega) \left[ \int v_{pr}(\tau) e^{i\omega(t-\tau)} d\tau \right] d\omega,\]  

which is generally more difficult to handle for a non-periodic \( v_{pr}(t) \) than eq. (6).

The above shows that it is sometimes simpler to study the behaviour of a system with reference to a discontinuous process (here the sudden application of a unit step voltage, giving the step-function response \( G_s(t) \)), than with reference to a continuous process (the continuous application of an alternating voltage \( \exp j\omega t \), from \( t = -\infty \) to \( t = +\infty \)). An objection to working with the unit-step function might be that, although mathematically easier to handle in such cases, true discontinuity cannot in reality be achieved. It should be remembered, however, that any discontinuous process can be regarded as the limit of a physically realizable process, and that, strictly speaking, there are in fact no perfectly continuous processes either. Van der Pol once remarked that there seemed to have been a long-standing aversion to the study of discontinuous phenomena. He described this aversion wittily as an "horror discontinuositatis", resembling the "horror vacui" assumed by 17th century physicists.

We have discussed these preliminaries to show how useful Heaviside’s operational methods can be for the study of electrical systems. This appears from the fact that eq. (6) — a so-called composition product or convolution integral, in this case of the two functions \( G_s(t) \) and \( v_{pr}'(t) \) — is particularly suitable for treatment by the methods of operational calculus. In the version of operational calculus adopted by Van der Pol and the author 18), to any arbitrary function \( h(t) \) (the "original") one may allot a new function, the operational "image" \( f(p) \), defined by the following Laplace integral:

\[
f(p) = p \int_{-\infty}^{+\infty} e^{-pt} h(t) \, dt.\]  

17) J. Haantjes, Philips tech. Rev. 6, 193, 1941.

18) B. van der Pol and H. Bremmer, Operational calculus, based on the two-sided Laplace integral, Cambridge Univ. Press 1950. — The version of the operational calculus developed in this book is characterized by the fact that the integral in eq. (8) has the lower limit \(-\infty \), as opposed to the more conventional definition used by Carson, Doetsch, Wagner et al., where the lower limit is always 0. The advantages of using \(-\infty \) as the lower limit are explained in the book.
The image of a convolution integral is simply the product of the images of the individual factors. In the case of eq. (6) we therefore find the operational image of the function \( v(t) \) as the product of the images of \( G_s(t) \) and \( v_{pr}'(t) \). By finding the original of this operational image we then have a complete solution of the problem.

The other expression for \( v(t) \), eq. (7), which must obviously be equivalent to eq. (6), has a form which does not lend itself so well to an operational solution. Since eq. (7) is precisely the form most suited for the treatment of periodic processes, we can see why the operational calculus is less useful for dealing with such processes than with transients. Nevertheless, eq. (7) must evidently also have some relation to operational calculus; this relation becomes clear if we recall that (7) represents a Fourier integral which, after appropriate substitutions, can be put in the form of the Laplace integral (8). It is precisely the linking of operational methods to an exponential integral of this kind (which is related to the familiar Fourier integrals) that has made this calculus into a rigourously grounded mathematical tool.

In order to apply operational methods to the solution of all kinds of concrete problems, such as determining the above-mentioned integral (6) with given explicit functions, it is evidently useful to have a list of images available that correspond to originals given by elementary functions. Van der Pol called such a list his "dictionary", whilst the rules of operational calculus were the "grammar". One of these rules, for example, is the foregoing theorem concerning the convolution integral.

Armed with such a grammar and dictionary Van der Pol studied numerous properties of electrical networks, and of filters in particular. He arrived at theorems, often of very general application, which concerned for instance the relation between the unit function response of a given filter and that of a new filter derived from it; the latter was produced from the original by replacing all inductances by capacitances, and vice versa. General theorems of this kind sometimes led to interesting special cases. One notable example was the discovery that the charging of a system of capacitors from a D.C. source always takes place with an efficiency of 50%, irrespective of the number of resistances and inductances in the circuit. This is to say that the final energy of the charged capacitors amounts to half the energy which the source, after being suddenly switched on, must deliver in order to sustain the currents which must flow until the system reaches its final state.

Many of the mathematical relations discovered by Van der Pol were arrived at quickly by the skilful manipulation of the methods of operational calculus. An example of such a relation in the case of continuous functions is the remarkable and previously unknown identity:

\[ a \int_0^\infty e^{-at} \sin^2 t \, dt = \frac{(2n)!}{(a^2+2^2) (a^2+4^2) \cdots (a^2+4n^2)} , \]

for \( \text{Re } a > 0 \). 

Van der Pol liked in particular to work on relations for discontinuous functions. A representative example is the function \( A_2(t) \), which indicates the number of pairs of integers \( (m,n) \) for which \( m^2 + n^2 < t \); operational methods led at once to the surprising relation:

\[ \frac{1}{2} \{ A_2(t) - 1 \} = \left[ \frac{t}{1} \right] - \left[ \frac{t}{3} \right] + \left[ \frac{t}{5} \right] - \left[ \frac{t}{7} \right] + \cdots , \]

where the symbol \( [\cdot] \) here represents the largest integer \( \leq \) the number within the brackets.

As a last typical example of the application of operational calculus we mention the "moving average" or "sliding mean" theory, which was extensively studied by Van der Pol. Starting from an arbitrary function \( g(t) \), the sliding mean is defined by the new function

\[ g^*(t) = \int_{t-A}^{t+A} g(\tau) \, d\tau . \]

In many cases we want to know the function \( g(t) \) when \( g^*(t) \) has been measured. Take, for example, the intensity \( g(\omega) \) as a function of frequency \( \omega \) in a spectrogram recorded with a spectrograph. Owing to the necessarily finite width of the slit in the spectrograph, the actual intensity \( g(\omega) \) is smeared out into a generally less-rapidly varying function \( g^*(\omega) \). The operational solution of this problem leads to many ways in which \( g \) can be expressed by a series in \( g^* \), suitable for numerical use.

Other subjects

In the foregoing we have discussed the most important general subjects on which Van der Pol was engaged. We shall now touch briefly on various other, more specialized problems on which he worked for shorter or longer periods — on some all his life —

---

and which will be found in the bibliography of his numerous publications 20).

Amongst the earliest researches of Van der Pol were special subjects concerned with the theory of electrical networks, including current distributions in an arbitrary number of coupled circuits. His interest in antenna theory was first aroused by investigations into the radiation and natural frequency of antennae possessing end capacitance, during which the inadequacy of the concept of radiation resistance was discussed. In England he carried out experimental research on the conductivity of seawater, in connection with its bearing on the propagation of radio waves. When he joined the Philips laboratory his interest extended to virtually all problems of radio technology, then still in its early stages. Partly in cooperation with K. Posthumus, Y. B. F. J. Groeneveld, T. J. Weijers, G. de Vries, C. J. Bakker, W. Nijenhuis, C. J. Bouwkamp and others (mentioned elsewhere in this article) he investigated the general properties of triode characteristics, the distribution of the electrical field and electron paths inside a triode, the theory of grid detection, general properties of oscillators and filters (including the equivalent circuit of a quartz oscillator, and the non-linear theory of hysteresis effects in two coupled circuits, one of which is part of a triode generator), noise in radio valves, radiation patterns of beam antennae, and many other subjects.

Van der Pol had an important share in the preparatory work leading to the introduction of radio broadcasting in the Netherlands. Together with R. Veldhuyzen, M. Ziegler and J. J. Zaalberg van Zelst he did extensive research into the field-strength distribution of broadcast waves over the Netherlands. The results were plotted on charts, showing contours of constant field strength. The deviation of these contours from a circular shape indicated the influence of the type of ground traversed between transmitter and receiver. In the chart for the Hilversum transmitter on 298.8 m wavelength, for example, the absorbent effect of the city of Amsterdam and of the dry sandy ground of the chain of hills east of Utrecht could be clearly seen. This chart led Van der Pol to point out that the field-strength distribution in the Northern provinces would change after the damming of the Zuiderzee, since the conductivity of this large expanse of water would decrease as it became progressively less salty. This was confirmed by later measurements.

When a new transmitter was planned to replace the old one at Hilversum, the reciprocity theorem was applied to determine its most favourable location: instead of starting with a transmitter in a central (though not yet determined) point, measurements were made of the fields due to two auxiliary transmitters in two distant points in the country, where reception from a centrally-situated transmitter would have been weakest. The most favourable site for the new transmitter was found by determining, in the central region of the country, at what point the received field was strongest on the line connecting the points of intersection of corresponding contours charted for both auxiliary transmitters (fig. 10a, b and fig. 11).

Long before there was any question of the practical exploitation of frequency modulation, which has since become so important in broadcasting, Van der Pol had investigated the theory of this system of modulation. Fundamental insight here requires much more mathematical knowledge than the simpler system of amplitude modulation, since the differential equations involved can no longer be solved by time functions proportional to exp jwt. In the first place, Van der Pol gave a suitable definition for the vague concept "instantaneous frequency", \( \omega_{\text{inst}} \). Writing the general frequency-modulated signal as:

\[
A \cos \varphi(t) = A \cos \left( \omega_0 t + m \omega_0 \int_0^t g(t) \, dt + \Phi \right),
\]

this definition reads:

\[
\omega_{\text{inst}}(t) = \frac{d}{dt} \varphi(t) = \omega_0 \left( 1 + m g(t) \right).
\]

Together with F. L. H. M. Stumpers he then investigated the so-called quasi-stationary approximation whereby the currents and voltages assume values at any given instant that conform to the then existing value of the instantaneous frequency as defined by (13). It was found that the current produced in an impedance as a result of a frequency-modulated voltage is then solely determined by the phase characteristic of the impedance 21). In the 'forties Van der Pol also carried out experimental research on frequency modulation, but lack of space must preclude its discussion here.

In the theoretical investigation of frequency modulation an important part was played by

\[20\) A complete bibliography of Van der Pol's publications, compiled by C. J. Bouwkamp, will appear shortly in Philips Res. Repts.

Fig. 10. Field-strength measurements carried out in 1934 in the central region of the Netherlands for determining the most favourable site for the new broadcasting station, planned at that time. In two remote parts of the country, where the poorest reception might be expected from a central transmitter, two identical auxiliary transmitters were erected, one at the Dollard estuary (wavelength 325 m), and the other at Maas- tricht (wavelength 317 m).

a) The field distribution of the Dollard transmitter represented by contours of constant field strength. The long arrow points to the transmitter.

b) The same for the Maastricht transmitter. (Note the shadow effect due to Rotterdam.)
Mathieu's differential equation. More generally, Van der Pol applied himself to the study of differential and difference equations related to radio problems. These studies again led to many and various investigations that had a less direct connection with physical or technical questions, or were even of a purely mathematical nature. They included theta functions, elliptic functions, the gamma function, the theory of numbers, the properties of prime numbers, and so on. Nevertheless, these problems cannot be seen as entirely distinct from radio technology. Theta functions, for example, are related to the theory of potential functions, and Van der Pol's interest in the theory of numbers was inspired by the part it plays, for instance, in the above-mentioned synchronization effects in relaxation oscillators. Incidentally, these abstract researches also led to a quite unexpected "industrial by-product", as a result of Van der Pol's study of the distribution of Gaussian prime numbers. In the system of all complex numbers with integral real and imaginary parts, these are the numbers that cannot be obtained as the product of two of them. When all these two-dimensional prime numbers are drawn in the complex plane, a curious pattern is produced (fig. 12). This was used as a tablecloth pattern which had particular success in America.

![Map of the Netherlands with field strength distribution](image)

**Fig. 11.** The bold lines connect the places where the two auxiliary transmitters are received with equal strength (thick central line, 1:1) or with an intensity ratio of 2:1 and 1:2. As can be seen, the highest field strength is found on the thick line at the point 18 near the village of Lopik (where in fact the station was subsequently built). The numbers at other places indicate the relative value of the field which, if the transmitter of the station had been sited there, would have been obtained in the more unfavourable of the two poor-reception areas (Dollard or Maastricht). (Fig. 10a, b and fig. 11 are taken from B. van der Pol, Rapport van de veldmetingen van twee bij de Dollard en bij Maastricht opgestelde proefzenders, T. Ned. Radiogen. 7, 173-195, 1935.)

The association of mathematics and radio technology sometimes worked in the opposite direction. We refer to an investigation whereby Van der Pol, in cooperation with C. C. J. Addink, used electrical techniques for studying a mathematical function. The investigation concerned the determination of the zero points of Riemann's \( \zeta \) function, normally defined by the series:

\[
\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \ldots \ldots \quad (14)
\]

Although this series can be used only for \( \text{Re} \, z > 1 \), the function can nevertheless be defined for the whole complex plane by means of an analytic continuation. Riemann had conjectured that the function thus defined would possess an infinite number of zero points, all of which would lie on the line \( \text{Re} \, z = \frac{1}{2} \).

By means of a relation derived by Van der Pol, the functional values along this line can be associated with the simple function:

\[
e^{-\frac{s}{2}} \left[ e^{s} - e^{-s} \right] \quad \ldots \ldots \quad (15)
\]

on a photocell placed behind it. The photo-current thus obtained, varying with time, was mixed with a sinusoidal current of frequency \( f \). The resultant signal, containing the difference tones between \( f \) and all harmonics of the periodic function derived from (15), was used to excite a mechanical resonator of very sharp resonance. When the frequency \( f \) was now slowly raised from zero, each successive harmonic of the above function excited the resonator in turn with its difference tone, and with an intensity corresponding to the relevant Fourier coeff-

![Diagram](image.png)

Fig. 12. The black squares represent in the complex plane (with origin in the centre) the complex prime numbers of Gauss, i.e. the numbers \((m + jn)\) that cannot be written as \(m + jn = (a + jb) (c + jd)\), where \(m, n, a, b, c, d\) are integers. This pattern was woven into a fabric marketed by an Eindhoven textile factory.

Musical theory

We shall now turn from Van der Pol's scientific and technical investigations and in conclusion touch briefly on his interest in music. Although deeply interested in analysing the structure of a musical work, he realized that this could never be an approach to the creative aesthetic element —
Fig. 13. Disc with edge serrations representing part of the function (15), for experimentally determining the zero points of Riemann’s $\zeta$ function.

an element for which, as a competent musician himself, he had a deep understanding. It was his preoccupation with the theory of numbers that led him to look for mathematical laws in harmonic

intervals $^{25}$), or, for example, to find numerical definitions for concepts such as dissonance. In particular he discovered that the fractions presented by the successive intervals in the diatonic scale could be contained in a Farey series. In this case it is the series of all irreducible fractions whose denominators and numerators never exceed the number 5. Only two tones from the diatonic sequence (the $b$ and the $d$, with $c$ as the fundamental) do not fit into the Farey series, and the remarkable thing is that it is precisely these tones about which there is most uncertainty in tuning (if one is not bound to the equal-temperament scale).

It will not be surprising after the foregoing that Van der Pol emphasised that the sub-harmonics of a given note can be produced with the aid of synchronization effects on relaxation oscillations. In this way it was possible to confirm experimentally a postulate put forward by the music theorist Hugo Riemann, namely that the minor triad can be regarded as the combination of the 4th, 5th and 6th sub-harmonics of a given tone, just as applies to the major triad in respect of the 4th, 5th and 6th upper harmonics of another tone.

$^{25}$) B. van der Pol, Muziek en elementaire getaltheorie, Arch. Mus. Teyler 9, 507-540, 1942 (in Dutch).

Fig. 14. Recording obtained with the aid of the disc in fig. 13, giving the Fourier coefficients of the function cut into the disc. The minima in the recording are the zero points of the $\zeta$ function on the line $\text{Re } z = \frac{1}{2}$. The first 29 zero points, denoted by dashes and Roman figures, were already known from calculations. (The imaginary part of $z$ is set out along the top edge.)
Van der Pol possessed absolute pitch, and was able, for example, to tune his violin to the right pitch without a tuning fork. He enjoyed exercising this gift and it was typical of the man that he should develop from it yet another interesting investigation. Together with Addink he devised an apparatus using an oscilloscope to check the tuning of an orchestra during a performance. From hundreds of observations, both on permanently tuned instruments and on orchestras, they found that the middle $a$ varied from 430 to 447 c/s, and also that there were quite distinct, more or less systematic variations of this frequency during a performance.

Van der Pol's enquiring mind led him to apply the principles of science to music. For him, however, there existed a much profounder relation between these fields. There is perhaps no more fitting way to end this review of the many-sided work of the great man of science that Van der Pol undoubtedly was, than to quote his own words, from a lecture given at the Teyler Foundation in Haarlem:

"Is there really any difference between the inspiration that leads to the conception of a beautifully flowing melody, a rich theme or a brilliant modulation, and that which leads to the conception of a new, elegant, unexpected mathematical relation or postulate? Are not both born in the same mysterious way and do they not both often demand laborious development? Art connotes skill, and skill too is the handmaid of science."

Summary. The review given deals principally with the work done by the late Van der Pol in the years from 1922 to 1949, when he was with the Philips Research Laboratories at Eindhoven. The main subjects discussed are the propagation of radio waves, especially the influence of the ground conditions on ground-wave propagation; non-linear circuit phenomena (in particular relaxation oscillations described by the "Van der Pol equation", and their synchronization); transients and operational calculus, which (following Heaviside, whom he greatly admired) Van der Pol used as a fruitful heuristic and later rigorously founded method. A brief discussion is devoted to various other subjects treated by Van der Pol in his numerous publications, including mathematical investigations prompted by problems of radio technology, such as the application of Mathieu's equation to frequency-modulation problems, and also studies relating to subjects of pure mathematics, such as elliptic functions and the theory of numbers (especially the properties of prime-numbers). Finally some examples are given of Van der Pol's interest in music and the theory of music.