The lateral skin effect in a flat conductor

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Introduction

The classical skin effect is a well-known phenomenon, a convenient reference being the recent survey by H. B. G. Casimir and J. Ubbink [1]. In a wire of circular cross-section, the radial distribution of the current density is a Bessel function of argument proportional to the square root of the frequency. At d.c., the density is uniform, whereas at high frequency the current is approximately concentrated in a peripheral sheet of thickness

\[ \delta = \sqrt{\frac{2}{\pi p \omega \sigma}} \]  

(\(p\) = permeability, \(\sigma\) = conductivity, \(\omega\) = angular frequency; practical electromagnetic units), called the skin depth. Correspondingly, the resistance \(R\) per unit length increases from the d.c. value \(R_0\), at first in proportion to the square of the frequency, but ultimately it tends to infinity as the inverse of the skin depth, and hence as the square root of the frequency.

Qualitatively, the preceding results hold for massive conductors of any cross-section with a smooth boundary, the only difference being that the tendency of the current density to concentrate towards the surface is more marked at the points where the curvature is greatest. For instance, in a conductor of elliptic cross-section, the density at the ends of the major axis will ultimately be larger than at the ends of the minor axis.

The effect is particularly marked when the eccentricity of the ellipse is very large, and an interesting problem is the limiting case of a very thin elliptic cylinder (fig. 1a), which almost reduces to a flat strip of width 2\(a\). The problem of major practical importance is, of course, the thin rectangular strip (fig. 1b) widely used in printed circuitry and in microwave applications. It so happens, however, that the elliptic case is amenable to analytical treatment whereas the rectangular case can only be treated numerically. Consequently both cases deserve to be considered, with the hope of deducing from the theory of the thin ellipse some results which hold, at least qualitatively, for the thin rectangle.

Because the skin-effect problem in thin conductors involves two dimensions of different orders of magnitude, a thickness 2\(b\), and a lateral dimension 2\(a\), with

\[ b \ll a, \]  

it automatically splits into two almost unrelated problems. At frequencies such that

\[ b \ll \delta \]  

the current density is still uniform along the thickness coordinate, so that the only problem is the lateral distribution of the linear current density (A/cm), to be written as \(i(x)\), along the width coordinate: \(i(x)dx\) thus designates the total current between \(x\) and \(x + dx\). The problem corresponding to the restriction (3) characterizes the lateral skin effect, and is the only one treated in this paper, because the depth penetration, occurring at much higher frequencies, is well known.

The plan of the article is as follows. A first section is devoted to a qualitative physical discussion of the lateral skin effect in thin conductors and the resulting increase in resistance with frequency. Although all the mathematical derivations are concentrated in the last section, some general remarks of a mathematical nature are necessary at the beginning and are included in the second section. A summary of the results is then presented: complete analytical results are given for the thin ellipse and are all original, to the best of our knowledge; the relatively meagre existing information on the thin rectangle is reviewed and a minor addition is made.

In the following section, the impedance (both for the ellipse and the rectangle) is characterized by its poles and zeros, which brings deeper additional information.
on its behaviour. Finally, an approximate treatment of the high-frequency behaviour of the impedance for the rectangle is given in the Appendix.

**Lateral skin effect**

As mentioned in the introduction, the skin effect in thin conductors can be separated into a lateral problem and a depth-penetration problem. The lateral problem is, however, of a different nature for conductors with no sharp edges (such as a hollow elliptic cylinder) on the one hand, and for conductors with sharp edges, on the other. Since condition (2) for a thin strip does lead to sharp edges (points of infinite curvature) at \( x = \pm a \), both for the thin ellipse and the thin rectangle, the current concentration towards the ends is more important, which results in a different law of increase of resistance. In this section we treat successively (a) thin conductors in general, (b) thin conductors with no sharp edges, (c) thin conductors with sharp edges.

Instead of the three dimensions \( a, b \) and \( \delta \) appearing in (2) and (3), it is convenient to introduce the two dimensionless ratios

\[
\frac{\gamma(ab)}{\delta} \quad \text{and} \quad \frac{a}{b}.
\]

The lateral problem is obtained when \( b/a \) is made strictly zero, as the limit of (2). On the other hand, at frequencies where \( \gamma(ab)/\delta \) becomes large compared to unity, the lateral skin effect reaches its asymptotic behaviour. This means that the linear density \( i(x) \) and the external electromagnetic fields have high-frequency limiting values which are independent of frequency. The corresponding behaviour will be called asymptotic lateral, and thus assumes a frequency range such that

\[
b \ll \delta \ll \gamma(ab),
\]

which is of course compatible with (2).

When the frequency increases further, so that (3) no longer holds, the true current density \( (A/cm^2) \) begins to vary along the thickness coordinate, but the linear density \( (A/cm) \) and the external fields keep their lateral asymptotic values, because of the separation noted earlier of the lateral problem from the depth-penetration problem.

For thin conductors, the pattern of the variation of resistance with frequency is markedly different from that for massive conductors. There are in fact two distinct phases of increase (the lateral effect and the depth effect), obeying different laws and separated by a large frequency interval corresponding to (5), where the lateral effect has already reached its asymptotic state while the depth penetration has not yet come into play. Moreover, the linear density and the external fields reach their asymptotic behaviour in the first phase and remain unaltered during the second phase.

For any thin conductor with no sharp edges, such as a thin hollow elliptic cylinder of moderate eccentricity, the asymptotic linear density is finite at every point, and so is the lateral asymptotic resistance. The law of resistance increase therefore has the form shown in fig. 2: the first lateral phase \( AB \) is followed by a long stationary interval \( BC \) corresponding to (5), where the resistance keeps its constant lateral asymptotic value, the second phase (depth penetration) is \( CD \) and the resistance ultimately increases as the square root of the frequency.

Although the problem of the thin hollow elliptic cylinder is rather academic, its lateral asymptotic behaviour (in the phase \( BC \)) is so elementary and illuminating that it deserves a short discussion. Since the external magnetic field satisfies Laplace’s equation and has no component normal to the ellipse, the lines of force are homofocal ellipses. The linear current density (along the periphery of the ellipse) is given directly by the discontinuity of the tangential component of the magnetic field and is thus inversely proportional to the distance along the normal between two adjacent ellipses of the family. In particular, the linear density is independent of the thickness \( 2h \) of the cylinder, even if the latter is variable. By contrast, the asymptotic resistance (the constant ordinate of \( BC \) of fig. 2) depends on the thickness, because the true density \( i/2h \) \((A/cm^2)\) in an element of area \( 2hds \) \((ds = \text{element of length along the boundary})\) produces a dissipated power

\[
\int (i/2h)^2 2h \, ds
\]

involving \( h \). In particular, if the thickness is chosen proportional to the lateral asymptotic density, \( i/2h \) is a constant in (6), just as at d.c., and the asymptotic value of \( R/R_0 \) is 1. Since the resistance is a non-decreasing function of frequency, it must remain constant in the whole lateral phase, for the law of thickness variation adopted. For the hollow elliptic cylinder, the corresponding law defines the conductor as the interior between two homothetic ellipses, which means that the ratios of the major and minor axes are equal. In such a conductor, the increase of linear current density towards the ends of the major axis is exactly compensated by the increased thickness, so that the true current density remains uniform. As a trivial particular case, there is no lateral skin effect in a hollow circular cylinder of constant thickness.

The above discussion, leading to the resistance behaviour of fig. 2, was specifically restricted to thin conductors with no sharp edges, and does not apply to the flat strip, whether rectangular or elliptic. This is because the lateral asymptotic linear current density
becomes infinite at the edges \((x = \pm a)\) of the strip, so that the dissipation \((6)\), and hence the resistance, is infinite for any thickness law \(h(x)\), unless \(h\) also becomes infinite at the edges, which is inconsistent with the assumption of a thin conductor. In fact the asymptotic linear density for a strip of any thickness is

\[
\frac{i(x)}{i_0} = \frac{2}{\pi \sqrt{1 - x^2/a^2}},
\]

where
\[
i_0 = I/2a
\]
is the average linear density, and \(I\) the total current. Expression \((7)\) is well known \([2]\) but will be derived again in this article. For a rectangular section, the linear density thus varies from the uniform density \((8)\) at d.c. to the asymptotic density \((7)\) at high frequencies. For the elliptic section, however, it is the true density \(i/2h\) which is uniform at d.c. Since the conductor of fig. 1 has the variable thickness \(2h(x)\), with

\[
h = b \sqrt{(1 - x^2/a^2)},
\]

the linear density at d.c. is not uniform, but is given by

\[
\frac{i(x)}{i_0} = \frac{4}{\pi} \sqrt{(1 - x^2/a^2)},
\]

so that the variation from d.c. to high frequencies in the elliptic case is much stronger.

Since the lateral asymptotic resistance of a flat strip is infinite, the law of resistance increase must be of the form qualitatively shown in fig. 3. At the end of the lateral phase \(AB\), the resistance reaches its asymptotic behaviour \(BCE\), from which it deviates in accordance with \(CD\) when depth penetration comes into play. Curve \(BCE\) tends to infinity in accordance with a law still to be discovered, but certainly more slowly than the square root of the frequency since it is dominated by the latter behaviour at the end of phase \(CD\). Finally, the lateral law \(AB\), and its asymptotic behaviour \(BCE\), are different for a thin rectangle and a thin ellipse, whereas the ultimate square-root law (at the end of phase \(CD\)) is the same in both cases.

Aspects of the mathematical treatment

In order to avoid certain duplications in the analysis of the elliptic and rectangular strips, we denote the thickness of the strip at the abscissa value \(x\) by \(2h(x)\), so that \(h\) is the constant \(b\) in the rectangular case, and the variable \((9)\) in the elliptic case. To permit a coherent normalized frequency to be used, both for the elliptic and the rectangular sections, we introduce the variable

\[
k = \frac{j\omega \mu}{4\mu R_0} = \frac{j\Omega}{\omega},
\]

which is proportional to the square of the first parameter \((4)\). We thus have the following notations:

<table>
<thead>
<tr>
<th></th>
<th>ellipse</th>
<th>rectangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.c. resistance</td>
<td>(R_0)</td>
<td>(\frac{1}{\pi \omega ab})</td>
</tr>
<tr>
<td>per unit length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normalized</td>
<td></td>
<td></td>
</tr>
<tr>
<td>frequency</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| The classical skin-effect equation for a massive conductor is
|                      |         |           | \(\Delta u + \lambda u = 0\), |
|                      |         |           | (13)       |
| with                |         |           | \(\lambda = - \frac{\mu}{\omega}\), |
|                     |         |           | (14)       |

where \( u \) stands for any axial component (electric field, current density, or vector potential) and where \( \Delta \) denotes the 2-dimensional Laplace operator. In the case of the circular cylinder, the magnetic lines of force are concentric circles at all frequencies, and there is no exchange of flux between the conductor and the surrounding dielectric. As a consequence, the internal current distribution can be studied alone, as a solution of (13) with circular symmetry, and the external field is usually disregarded. By contrast, for elliptic and rectangular sections, the magnetic field has a non-zero normal component penetrating into the conductor (except at high frequencies), so that the internal problem is not separable from the external one. Since \( q = 0 \) in the dielectric, the external problem satisfies (13) with \( \lambda = 0 \), which is a Laplace equation, and the external and internal solutions are connected by boundary conditions. For thin conductors, only the Laplace equation and the boundary conditions remain, which produces a considerable simplification.

At a large distance \( D \) from its centre, a conductor carrying a total current \( I \) produces a magnetic field of tangential component \( I/2\pi D \), and hence a magnetic flux per unit length proportional to \( I/nD \), which tends to infinity with \( D \).

Since the external problem depends on the position of the return conductor, which cannot be relegated to infinity, because of the preceding difficulty a coaxial return conductor is generally assumed of large, but finite, radius \( D \), concentric with the go conductor. This makes negligible the proximity effect of the return conductor, but the arbitrary constant \( D \) appears in all inductance expressions.

Any skin-effect impedance \( Z = R + joL \) has the nature of the impedance of a (distributed) RL circuit. In particular, it is known from circuit theory that the inductance \( L \) is a monotonically decreasing function of the frequency and takes its minimum value \( L_\infty \) at infinity. Since there is no internal magnetic field at high frequencies, \( L_\infty \) is the only natural definition of the external inductance. Moreover, \( Z = joL_\infty \) is then a minimum-reactance impedance, and the constant \( D \) disappears in this difference. In the following, we always evaluate the reduced impedance

\[
Z = \frac{Z - joL_\infty}{R_0},
\]

where \( R_0 \) is the d.c. resistance (12). The external inductance \( L_\infty \) is the one related to the capacitance \( C \) per unit length of the pair of conductors by \( CL_\infty = 1/c^2 \), where \( c \) is the velocity of light. For a thin strip (ellipse or rectangle) we have:

\[
L_\infty = \frac{\mu}{4\pi} \ln \frac{2D}{a}.
\]

**Summary of results**

**Thin elliptic strip**

For the thin elliptic strip, the normalized impedance (15) is

\[
z = \frac{J_k(k)}{J_k'(k)},
\]

where \( J_k(k) \) is the Bessel function of the first kind of order and argument \( k \) given by (11). In the denominator of (17), \( J' \) denotes the derivative with respect to the argument (and not to the order), and hence the value of \( dJ_k(s)/ds \) at \( s = k \).

In terms of the auxiliary variable

\[
u = \arccos x/a,
\]

the linear current density is given by any of the three following equivalent expressions:

\[
i = \frac{2}{\pi \sin \nu} \left[ 1 - \frac{2}{k J_k'(k)} \sum n J_{n+k}(k) \cos 2nu \right],
\]

\[
i = \frac{4\sin \nu}{\pi J_k'(k)} \left[ J_k(k) + 2 \sum J_{n+k}(k) \cos 2nu \right],
\]

\[
i = \frac{4}{\pi J_k'(k)} \sum \left[ J_{k+n-1}(k) - J_{k+n}(k) \right] \sin (2n-1)u,
\]

where all sums are for \( n = 1, 2, \ldots, \infty \) and where \( i_0 \) is (8). For \( k \) infinite, (19) reduces to its first term and gives (7), by (18). The series (19) is, however, divergent at the end-points \( x = \pm a \) corresponding to \( u = 0 \) or \( \pi \). By contrast, (20) and (21) are convergent, and (20) reduces to (10) for \( k = 0 \).

A continued-fraction expansion of (17) is

\[
z = \frac{1}{2} + \frac{1}{k + \frac{1}{1 + \frac{1}{k + \frac{1}{k + \ldots}}}},
\]

where the successive denominators are alternately 1 and \( 2n/k \) \((n = 1, 2, \ldots)\). The impedance (17) is thus represented by the equivalent circuit of fig. 4. The approximation of (22) limited to order \( k^2 \) is

\[
z = 1 + \frac{k}{2} - \frac{k^2}{4}.
\]

At high frequencies, the known asymptotic expression (8) of (17) is

\[
z = C k^{1/3},
\]
where the numerical factor $C$ is given by:

$$C = \frac{\Gamma(1/3)}{2^{1/3} \Gamma(2/3)} = 1.088 \ldots \quad (25)$$

With the definition (11) of $\Omega$, we thus obtain:

$$z = (0.942 \ldots + j 0.544 \ldots)\Omega^{1/3} \quad (26)$$

The real and imaginary parts of the function $z(k)$ are shown in Fig. 5 and compared with the tangents resulting from (23) and with the asymptotic expression (26). The numerical computation of (17) was based on the expansion (22) with truncations corresponding to 20 or 30 $RL$ sections in the equivalent circuit of Fig. 4; this produced no significant difference in the results, in the range $|k| < 10$.

**Thin rectangular strip**

In comparison with the full analytical results just summarized for the thin ellipse, very little is known for the thin rectangle. From a numerical study of the integral equation for the linear current density, V. Bezlevitch et al. [4] have obtained low-frequency approximations for the impedance. On the other hand, a purely numerical treatment of the problem (by different methods) has led P. Silvester [5] and C. Beccari and C. Ronca [6] to a resistance law confirming the earlier measurements of A. E. Kennelly and H. A. Affel [7]. From all these results it appears that the relative resistance increase $\text{Re}(z-1)$ for the ellipse is about twice that for the rectangle. The only new result obtained in this article is an analytic expression of the rectangle resistance as the ratio of two infinite determinants:

$$z = \frac{A_{11}}{A}, \quad (27)$$

where $A$ is the determinant of the symmetric matrix

$$\begin{vmatrix}
1 & -2 & -2 & -2 & \cdots \\
-2 & 2 & 1 & 1 & \cdots \\
-2 & -2 & 2 & 1 & \cdots \\
-2 & -2 & -2 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix}, \quad (28)$$

and $A_{11}$ the principal minor separated by dotted lines. In the matrix elements, the first term is constant on a parallel to the main diagonal and the second on a parallel to the second diagonal. Results (27)-(28) are discussed further in the next section and in the Appendix.

**Poles and zeros**

An impedance $z(k)$ as defined by (15) satisfies

$$z(0) = 1, \quad (29)$$

and is an $RL$ impedance, so that its zeros $k_1, k_2, \ldots$ and its poles $k_1', k_2', \ldots$ are negative real and separate each other:

$$0 < -k_1 < -k_1' < -k_2 < -k_2' < \ldots \quad (30)$$

The present section is devoted to an analytical and numerical study of the distribution of the poles and zeros, both for the ellipse and for the rectangle, with the idea of obtaining additional information on the impedance behaviour of the rectangle and, more particularly, on its asymptotic behaviour.

Physically, poles and zeros characterize transient modes of decay (with different boundary conditions)

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and correspond to free solutions of the skin-effect equation, hence to eigenvalues $\lambda$ of the Laplace operator in (13). The asymptotic distribution of these eigenvalues has been studied in mathematical physics in connection with various problems of statistical thermodynamics; in particular, the $n$th eigenvalue $\lambda_n$ of (13), for a two-dimensional domain of area $S$, and for quite general boundary conditions is known [1] to be asymptotic to $4\pi n^2/S$. From (14), (11) and $R_0 = 1/\rho S$, we thus obtain $-n$ as the asymptotic value of the $n$th pole and zero, which means:

$$k_n = -n - \alpha_n; \quad k'_n = -n - \beta_n \quad (n = 1, 2, \ldots, \infty), \quad (31)$$

with $\alpha_n/n$ and $\beta_n/n$ tending to zero for large $n$. Moreover, the alternation (30) of poles and zeros restricts the deviations of (31) to

$$0 < \beta_n - \alpha_n < 1. \quad (32)$$

Although the asymptotic distribution (31) is independent of the shape of the section so that it also holds for a circular cross-section of radius $a$, not all natural frequencies are excited by the forced current in this case (by circular symmetry), so that only a small subset of the eigenvalues (corresponding to a one-dimensional problem where $\lambda_n$ is asymptotic to $n^2 \pi^2/a^2$) appear as poles and zeros of the impedance.

Since $\alpha_n/n$ and $\beta_n/n$ certainly tend to zero if $\alpha_n$ and $\beta_n$ tend to constant values, it is interesting to discuss the case where $\alpha_n$ and $\beta_n$ are rigorously constant in (31). The function satisfying (29) is then

$$\frac{I(\alpha + 1)}{I(\beta + 1)} \frac{I(k + \beta + 1)}{I(k + \alpha + 1)} \quad (33)$$

and is asymptotic to

$$\frac{I(\alpha + 1)}{I(\beta + 1)} k^{\beta - \alpha} \quad (34)$$

for large $|k|$, by Stirling’s approximation for the gamma function.

If $\alpha_n$ and $\beta_n$ are not constant but tend sufficiently quickly to constant values $\alpha$ and $\beta$, the asymptotic expression of the impedance is still of the form $Ck^{\beta - \alpha}$, but with a coefficient $C$ different from that of (34), because the latter was imposed on (33) by condition (29). The asymptotic law may, however, become quite different when the variation of $\alpha_n$ or $\beta_n$ is very slow, as shown by the example of the logarithmic derivative $\Psi(k) = I'(k)/I(k)$ of the gamma function. For the function $\Psi(k + 1)$ we have $\beta_n = 0$, whereas $\alpha_n$ tends to zero [1] as $-\ln n$. Although (34) gives the value $1$ for $\beta - \alpha = 0$, the $\Psi$ function tends to infinity as $\ln k$.

For the ellipse impedance (17), it is known [10] that the $n$th zero of $J_0(k)$ is asymptotic to $-n + \frac{1}{2}$. Since the exponent $\beta - \alpha$ of (34) is known to be $\frac{1}{2}$ in (24), one thus expects the $n$th pole to be asymptotic to $-n + \frac{1}{2}$. This is confirmed by Table I based on a numerical computation for the equivalent circuit of fig. 4 with 20 and 30 sections.

### Table I. Zeros ($k_n$) and poles ($k'_n$) of the function $J_0(k)/J_1'(k)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k_n$</th>
<th>$k'_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.83732</td>
<td>-1.17172</td>
</tr>
<tr>
<td>2</td>
<td>-1.83409</td>
<td>-2.13294</td>
</tr>
<tr>
<td>3</td>
<td>-2.83422</td>
<td>-3.14014</td>
</tr>
<tr>
<td>4</td>
<td>-3.83393</td>
<td>-4.14441</td>
</tr>
<tr>
<td>5</td>
<td>-4.83377</td>
<td>-5.14728</td>
</tr>
<tr>
<td>6</td>
<td>-5.83368</td>
<td>-6.14937</td>
</tr>
</tbody>
</table>

### Table II. Zeros ($k_n$) and poles ($k'_n$) of the impedance of a thin rectangular conductor.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k_n$</th>
<th>$k'_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.87914</td>
<td>-0.9716</td>
</tr>
<tr>
<td>2</td>
<td>-2.06221</td>
<td>-2.1591</td>
</tr>
<tr>
<td>3</td>
<td>-3.10107</td>
<td>-3.2156</td>
</tr>
<tr>
<td>4</td>
<td>-4.12081</td>
<td>-4.2496</td>
</tr>
<tr>
<td>5</td>
<td>-5.13360</td>
<td>-5.2742</td>
</tr>
<tr>
<td>6</td>
<td>-6.14298</td>
<td>-6.2935</td>
</tr>
<tr>
<td>7</td>
<td>-7.1504</td>
<td>-7.3093</td>
</tr>
<tr>
<td>8</td>
<td>-8.1565</td>
<td>-8.3227</td>
</tr>
<tr>
<td>9</td>
<td>-9.1618</td>
<td>-9.3341</td>
</tr>
<tr>
<td>10</td>
<td>-10.1664</td>
<td>-10.344</td>
</tr>
<tr>
<td>11</td>
<td>-11.1705</td>
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</tr>
<tr>
<td>12</td>
<td>-12.1743</td>
<td>-12.358</td>
</tr>
<tr>
<td>14</td>
<td>-14.1809</td>
<td>-14.365</td>
</tr>
<tr>
<td>15</td>
<td>-15.1838</td>
<td>-15.365</td>
</tr>
</tbody>
</table>

For the rectangle, the zeros-and poles have been computed from (27)-(28) on matrices truncated at orders 30 and 40, and are given in Table II which confirms the asymptotic behaviour (31). This behaviour was also apparent in Silvester’s results [3] in spite of differences in normalization in (11) and (15): Silvester’s rough tabulation gives the zeros (but not the poles) of $z + k$ in $2$ in terms of the frequency variable $4k$, but is coherent with our results.

Although the natural frequencies have similar asymptotic distributions for the rectangle and the ellipse, the deviations $a_n$ and $b_n$ show a markedly different small-scale behaviour, as it appears in fig. 6. In contrast with the rapid convergence to the asymptotes for the ellipse, the deviations show a logarithmic drift for the rect-
angle, with an increasing difference $\beta_n - \alpha_n$. Since (32) prevents such an increase from continuing indefinitely, we are far from having reached the asymptotic behaviour of the deviations. Since, however, a continuation of the logarithmic drifts with the slopes resulting from fig. 6 is compatible with (32) up to about $n = 10^8$, there is little hope of obtaining the true asymptotic behaviour by numerical computations.

Our success in obtaining the asymptotic expression (24) of the ellipse impedance is due to the existence of the closed-form expression (17) and to the availability of relatively advanced Bessel-function data. If it were possible to establish (24) directly, without using the closed form (17), a similar approach might succeed for the rectangle where such a form is not available. Such a direct method is described in the next paragraphs for the ellipse. For the rectangle, the relevant mathematics are much more complicated and the treatment is given in the Appendix.

The asymptotic impedance of the ellipse will now be obtained by simple physical considerations of the network of fig. 4, which are, of course, equivalent to mathematical considerations of the corresponding continued fraction (22). A finite approximation of degree $n$ of the network is obtained either by short-circuiting the $(n + 1)$th shunt inductance or by opening the $(n + 1)$th series resistance, and the corresponding impedances will be called $Z_s$ and $Z_0$. At high frequencies, the network reduces to $n$ unit resistances in series in the first case, so that the approximate impedance is

$$Z_n = n.$$  \hspace{1cm} (35)

In the second case, however, the impedance of the last inductance dominates the last resistance at high frequencies, and the resistance can be neglected; the last inductance thus combines in parallel with the preceding one, and the reasoning applies again. Ultimately, the network reduces to the parallel combination of the first $n$ inductances. Since the total susceptance is $2(1 + 2 + 3 + \ldots + n)$, which is approximately $n^2$ for large $n$, we obtain the impedance

$$Z_0 = k/n^2.$$  \hspace{1cm} (36)

In classical network and line theory, the input impedances $Z_0$ and $Z_s$ of a dissipative 2-port opened or shorted at its output converge to a common value $Z$ (the characteristic impedance) when the network attenuation, or the line length, becomes infinite. By contrast, the divergent behaviour of (35)-(36) is due to the essential singularity at infinity of the function (17), resulting from its asymptotic behaviour (24). Although all three impedances $Z_0$, $Z_s$ and $Z$ thus diverge for large $|k|$, there is some hope of obtaining the asymptotic expression of $Z$ by imposing a common asymptotic behaviour on $Z_0$ and $Z_s$. As first attempt, if $Z_0/Z_s = 1$ is forced into (35)-(36), the resulting constraint

$$n^3/k = 1$$  \hspace{1cm} (37)

imposes the common value $k^{1/3}$ on $Z_0$ and $Z_s$, so that the asymptotic law (24) is confirmed, except for a small difference in the coefficient $C$, whose correct value (25) is replaced by 1. The discrepancy is due to the fact that the principal values (35)-(36) of $Z_s$ and $Z_0$ have been computed by making $k$ large for a fixed value of $n$, without considering the constraint (37) which was only found a posteriori, and as a first approximation. This suggests that an iterative process might lead to an improvement both of the constraint (37) and of the resulting value of the coefficient $C$ of (24). The mathematical justification of this process is based on the inequalities $Z_s < Z < Z_0$, holding for any positive $n$ and $k$ because of potentiometric effects, which impose a common asymptotic behaviour on all three impedances if $Z_0/Z_s$ is forced to tend to 1. Owing to the divergent values (35)-(36) of $Z_s$ and $Z_0$ for large $k$, fixed $n$, the ratio $Z_0/Z_s$ can only tend to 1 if $n$ and $k$ tend simultaneously to infinity, in accordance with some (as yet unknown) constraint. Because the constraint is unknown, it can only be reached by successive approximations, which lead to improved estimates for the asymptotic expression of $Z$ because the margins resulting from the potentiometric inequalities are decreased at every step. Although the process has not been proved convergent, the margins obtained in one or two steps are already sufficiently small for all practical purposes.

The second approximation replacing (35)-(36)-(37) is obtained as follows. When a unit current is injected in the network of impedance (35), the input voltage is $n$ and the voltage at the $i$th mode is proportional to $n - i$. The total magnetic energy in the shunt inductances is thus

$$\frac{1}{2} \sum_{i=1}^{n} 2(n - i)^2 \approx \frac{n^4}{12},$$

and must be equated to the energy $n^2/2L$ in an equivalent inductance $L$ shunting the resistance (35). Since $u = n$, we obtain $L = 6/n^2$, and (35) is replaced more accurately by

$$Z_s = \frac{n}{1 + n^2/(6k)}.$$  \hspace{1cm} (38)

By a similar reasoning we are led to evaluate the total dissipation in the network of the initially reactive impedance (36) and to represent it as a series resistance, which is found to be $8n/15$, so that (36) is replaced by

$$Z_0 = \frac{k}{n^2} \left(1 + \frac{8n^2}{15k}\right).$$  \hspace{1cm} (39)

It is not legitimate to equate (38) and (39) rigorously, for the re-
sulting second-degree equation contains terms in \((n^2/k)^2\) that are still neglected in (38)-(39). One must therefore linearize the equation around the first-order approximation (37), by replacing \(n^3/k\) by 1 in the correction factors. This reduces (38) to \(6n/7\) and (39) to \(23k/15n^2\). By equating the last two expressions we obtain for both the value (24) with \(C = 1.044\).

**Mathematical derivation**

The mathematical formulation of the lateral skin-effect problem for a thin conductor will now be derived simultaneously for the elliptic section (fig. la) and for the rectangular section (fig. 1b) with the common notation \(2h\) for the thickness, \(1\) being given by (9) in the first case and equal to the constant \(b\) in the second. The linear current density is the discontinuity of the tangential component \(H_z\) of the magnetic field. By symmetry we have for a right-handed coordinate system:

\[ i = -2H_z \bigg|_{y=+0} \]  

(40)

On the other hand, the true current density (A/cm²) is

\[ \frac{i}{2h} = q E_z. \]  

(41)

Finally, Lenz’s law yields:

\[ \frac{\partial E_z}{\partial x} = \mu j \omega H_y. \]  

(42)

It is convenient to introduce the vector potential \(A\) which has only a \(z\)-component. We then have:

\[ H_z = \frac{1}{\mu} \frac{\partial A_z}{\partial y}; \quad H_y = -\frac{1}{\mu} \frac{\partial A_z}{\partial x}, \]  

(43)

whereas the voltage drop \(\mathcal{Z}I\) along the conductor is

\[ -\frac{\partial V}{\partial z} = E_z + j\omega A_z. \]  

(44)

In (44), \(E_z\) is expressed in terms of \(A_z\) by (41), (40) and the first equation (43); this yields:

\[ \mathcal{Z}I = j\omega A_z - \frac{1}{\mu \Omega h} \frac{\partial A_z}{\partial y} \bigg|_{y=+0}. \]  

(45)

On the other hand, by elimination from (40)-(41)-(42), we obtain the Biot boundary condition [11]:

\[ \frac{\partial}{\partial x} \left( \frac{H_z}{\Omega h} \bigg|_{y=+0} \right) = -\mu j \omega H_y. \]  

(46)

When the magnetic fields are eliminated from (46) by (43), the resulting relation shows that the derivative of (45) with respect to \(x\) is zero. Biot’s condition is thus equivalent to saying that the impedance computed by (45) is independent of \(x\). The form (45) of the boundary condition is to be preferred to (46), since it yields the impedance without additional effort. Finally, the problem amounts to solving the Laplace equation for \(A_z\), with the condition (45) on the conductor (i.e. for \(y = 0, |x| < a\)), and the prescription of the value

\[ A_z = -\frac{I_\mu}{2\pi} \ln \frac{\sqrt{x^2 + y^2}}{D} \]  

(47)

at large distance, corresponding to the vector potential of a filament of current \(I\) at the origin, with a coaxial return of large radius \(D\).

We introduce the conformal transformation

\[ x + jy = a \cos (u + jv), \]  

(48)

yielding

\[ x = a \cosh v \cos u; \quad y = a \sinh v \sin u. \]  

(49)

The transformation is one-to-one with the restrictions

\[ -\pi \leq u \leq \pi; \quad v \geq 0. \]

In the \((x,y)\)-plane, the curves of constant \(v\) are homofocal ellipses (fig. 7); the segment \(y = 0, |x| < a\) is the infinitely flat ellipse \(v = 0\), and \(v\) increases outwards to infinity. The curves of constant \(u\) are the hyperbolae of fig. 7 but there is a cut along the segment \(v = 0\), so that \(u\) is positive in the upper half-plane \(\text{Re} y > 0\) and negative in the lower half-plane. The semi-infinite segment \(y = 0, x \geq a\) corresponds to \(u = 0\), whereas the segment \(y = 0, x \leq -a\) corresponds to \(u = \pm \pi\).

[Fig. 7. The conformal representation \(x + jy = a \cos (u + jv)\).]

The expression of \(A_z\) is of the form

\[ A_z = \frac{I_\mu}{2\pi} \left[ \ln \frac{2D}{a} - v + \sum A_n e^{-2\pi v} \cos 2nu \right], \]  

(50)

where the first two terms give the principal value (47), because (48) yields

\[ \frac{f(x^2 + y^2)}{D} = \frac{a \cosh v}{D} \approx \frac{ae^v}{2D}, \]

whereas the sum (extending from \(n = 1\) to \(\infty\)) with
undetermined coefficients \( A_n \) is the general harmonic function, vanishing at infinity, and having the appropriate quadrantal symmetry.

For \( y = +0 \), and hence \( v = 0, u > 0 \), then by (49):
\[
\frac{\partial A_z}{\partial y} \bigg|_{y=+0} = \frac{1}{a |\sin u|} \frac{\partial A_z}{\partial v} \bigg|_{v=0},
\]
so that (45) becomes:
\[
Z = \frac{j \omega \mu}{2\pi} \left( \ln \frac{2D}{a} + \sum A_n \cos 2nu \right) + \\
+ \frac{1}{2\pi \mu |\sin u|} (1 + 2 \sum n A_n \cos 2nu).
\]

The coefficients \( A_n \) must now be determined so that (51) takes a constant value for all \( u (\pi \leq u \leq \pi) \).

For the elliptic section, the first relation (49) reduces to (18) for \( v = 0 \), and (9) becomes
\[
h = b |\sin u|,
\]
so that the denominator in (51) simplifies to
\[
h |\sin u| = b \sin^2 u = \frac{b}{2} (1 - \cos 2u).
\]

With the notations (15)-(16) and (12), and with the substitution
\[
B_n = -k A_n,
\]
(51) multiplied by (53) becomes
\[
(z + 2 \sum B_n \cos 2nu) (1 - \cos 2u) + \\
+ \frac{2}{k} \sum n B_n \cos 2nu = 1.
\]

Replacing the product of cosines occurring in the left-hand side of (55) by cosines of sums and differences, one obtains a Fourier series, and its identification with \( 1 \) yields the infinite system
\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & \ldots \\
-1 & 2(1+1/k) & -1 & 0 & 0 & \ldots \\
0 & -1 & 2(1+2/k) & -1 & 0 & \ldots \\
0 & 0 & -1 & 2(1+3/k) & -1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix}
\begin{bmatrix}
z \\
B_1 \\
B_2 \\
B_3 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

of linear equations in \( z \) and the unknown coefficients \( B_n \).

Disregarding temporarily the first two equations contained in (56), one obtains the three-term recurrence relation
\[
B_{n-1} + B_{n+1} = 2(1 + n/k)B_n \quad (n = 2, 3, \ldots), \quad (57)
\]
which is very similar to the recurrence relation
\[
J_{v-1}(s) + J_{v+1}(s) = \frac{2v}{s} J_v(s) \quad (58)
\]
for Bessel functions. For \( v = n + k, s = k \), (58) shows that
\[
B_n = C J_{n+k}(k) \quad (59)
\]
satisfies (57), with \( C \) an arbitrary constant. In fact (59) is the solution of the second-order difference equation (57) in our case, because the other linearly independent solution (involving a Bessel function of the second kind) is excluded on physical grounds since it makes all coefficients \( B_n \) infinite at d.c. Since (57) holds down to \( n = 2 \) and thus involves \( B_1 \), (59) holds down to \( n = 1 \) and only two unknowns remain: the common factor \( C \) of (59), and the impedance \( z \). These are determined by the first two equations (56) which have been disregarded. By solving these equations, and making use of the known expression for the derivative of a Bessel function:
\[
J_v'(s) = -J_v(s) - J_{v+1}(s),
\]
(used for \( v = s = k \), we obtain
\[
C = \frac{1}{J_k'(k)}, \quad (61)
\]
and (17).

From the known expression (50) for \( A_z \), where \( A_n \) is deduced from (54), (59) and (61), we can compute \( H_z \) by (43) and \( i \) by (40). This gives (19). The other form (20) is obtained when \( i \) is computed by (41) with the value of \( E_z \) deduced from (44) where \( \frac{\partial V}{\partial z} \) is \( ZI \). This completes the proof of all the basic results, (17) to (21), for the ellipse.

The expansion (22) of (17) can be deduced from the three-term recurrence relations (57). In the equivalent network of fig. 4, they correspond to the Kirchhoff relations between the currents in branches incident to a common node, and the following electrical proof is therefore equivalent to a mathematical discussion of (57). Consider the ladder network of fig. 8, where the series admittances are denoted \( Y_1, Y_2, \ldots \) and the shunt admittances...
admittances $Y_a, Y_b, \ldots$. Denote by $V_1$ the node potentials with respect to ground, as indicated. If a unit current is injected at the input, elementary node analysis yields the linear system

$$
\begin{bmatrix}
Y_1 & -Y_1 & 0 & 0 & \ldots \\
-Y_1 & Y_1+Y_a+Y_2 & -Y_2 & 0 & \ldots \\
0 & -Y_2 & Y_2+Y_b+Y_3-Y_3 & \ldots \\
\vdots & & & & \ddots
\end{bmatrix}
\begin{bmatrix}
V_0 \\
V_1 \\
V_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}
$$

(62)

for the node voltages, and the solution $V_0$ is the input impedance. From a comparison of (56) and (62), it is found that the solution $x$ of (56) is the input impedance of the ladder network of fig. 4 where the elements are normalized ($k$ is taken as the complex frequency). Since the input impedance of fig. 4 is (22), we have indirectly obtained the continued-fraction expansion of the function (17).

For the rectangular section, $h$ is constant in the denominator of (51). After multiplication by the known Fourier series

$$
\sin u = \frac{4}{\pi} \left( \frac{1}{2} - \frac{\cos 2u}{1 \times 3} - \frac{\cos 4u}{3 \times 5} - \ldots \right)
$$

(63)

and substitution of (54), (51) becomes

$$(x + 2 \sum B_n \cos 2nu)(1 - \frac{2}{1 \times 3} \cos 2u - \frac{2}{3 \times 5} \cos 4u - \ldots) + \frac{2}{k} \sum B_n \cos 2nu = 1.$$  

(64)

The linear system resulting from (64) is analogous to (56) except that the matrix is now (28). This establishes (27).

**Appendix: High-frequency impedance of the rectangular strip**

At the end of the section on poles and zeros, we succeeded in obtaining a good approximation of the asymptotic impedance of the ellipse, without using its closed-form expression (17). The result was (24) but, instead of the correct coefficient 1.088 . . . of (25), we obtained 1 and 1.044 . . . by successive approximations. In this Appendix, we apply the same method to the rectangle.

The analysis is, however, much more difficult, essentially because the matrix (28) is a full matrix whereas the one (56) for the ellipse was tridiagonal, so that only the first approximation will now be worked out. Consequently, the accuracy of the result cannot be assessed. Also, Silvester’s numerical results cover too narrow a frequency range to provide an adequate verification. In spite of its limited significance, our result is the only one presently available; it can be improved by further research, and has been obtained by a method having its own mathematical interest.

As for the ellipse, the following analysis is based on the equivalent circuit, but could be translated into purely mathematical terms. Because the matrix (28) is not tridiagonal, the circuit is not a ladder network, but a general $RL$ network with an infinite number of nodes. Since, however, terms in $k^{-1}$ only occur on the diagonal, admittances only connect each node to ground and the interconnections between non-ground nodes are purely resistive. Finally, for $k = \infty$, the sum of all elements in any row of (28) is zero, on account of (63) for $u = 0$. This means that the direct conductance from any node to ground is zero, so that the branch connecting node $n$ to ground is the admittance of value $1/2nu$ alone, as in fig. 4.

When the network is truncated at $n$ nodes, by opening all resistances leading to further nodes, what remains is an $n$-node resistive network with an admittance from each node to ground, but with no resistive path from the first node to ground. The impedance from node 0 to ground is thus infinite at high frequency and its principal value is due to the admittances alone, so that all resistances can equally be short-circuited. The resulting impedance is the parallel combination of the first $n$ admittances, and this yields (36), as in the elliptic case.

In the complementary method of truncation at $n$ nodes, all further nodes are shorted to ground. The resistances leading to further nodes then produce a resistive path from node 0 to ground, so that the impedance is resistive at high frequency, as in (35), and can be evaluated by open-circuiting all the admittances. This amounts to computing (27) for the matrix (28) truncated at order $n$ and for $k = \infty$. Note that the sum of the elements in each row is no longer zero, owing to the truncation, so that the matrix is non-singular. Finally, the evaluation of the truncated impedance is equivalent to solving (64) for $x$ with

$$
k = \infty, B_{n+1} = B_{n+2} = \ldots = 0 .
$$

(65)

Note that the truncation of the Fourier series of coefficients $B_i$ corresponds to the shorting of the higher nodes, whereas the Fourier series (63) appearing in (64) is not truncated, for this corresponds to the preserved resistive connections to higher nodes. By (63), the form of (64) modified by (65) is thus

$$
z_s + 2 \sum_{i=1}^{n} B_i \cos 2nu = \frac{2}{\pi} \sin u .
$$

(66)

In the original (non-truncated) form (64), originating from the boundary condition (45), the coefficients $B_i$ and the impedance $x$ were determined by making the latter independent of $u$. Because of the truncation, this becomes impossible rigorously, and $z_s$ can only be made constant at $n$ points (the number of undetermined coefficients) and, owing to the quadrantal symmetry of (66), these may all be chosen in one quadrant, say the first. The quadrantal symmetry is preserved by choosing $n$ equidistant points with intervals $\pi/2nu$ from each other and half that interval from the ends. The interpolation points are thus

$$
u = \frac{\pi}{4n}, \frac{3\pi}{4n}, \frac{5\pi}{4n}, \ldots, \frac{(2n-1)\pi}{4n} .
$$

For this classical trigonometric interpolation, the “d.c. component” $z_s$ of (66) is simply the mean value of the second term of (66) at the interpolation points:

$$
z_s = \frac{1}{n^2} \sum_{i=0}^{n-1} \frac{1}{\sin (2i + 1)\pi/4n} .
$$

(67)

Since the above derivation of (67) is rather indirect, we check that the same derivation yields the known value (35) in the elliptic case. Equation (55) with the reductions (65) then yields

$$
z_s + 2 \sum_{i=1}^{n} B_i \cos 2nu = \frac{1}{1 - \cos 2u} = \frac{1}{2 \sin^2 u} .
$$

and (67) is replaced by

$$
z_s = \frac{1}{2n} \sum_{i=0}^{n-1} \frac{1}{\sin^2 (2i + 1)\pi/4n} ,
$$

which is indeed (35), by a known identity.
No closed-form expression is available for (67), but, for large \( n \), it can be approximately evaluated with the help of the Euler-MacLaurin summation formula applied to the function

\[
\frac{1}{\sin x} - \frac{1}{x}
\]

so as to extract the singularity at \( x = 0 \). In this way, (67) is approximately obtained as

\[
z_s = \frac{4}{\pi^3} \ln \frac{16n\gamma}{\pi}
\]

where \( \gamma = 1.781 \ldots \) is Euler’s constant.

By identifying (36) and (69), and eliminating \( n \), one obtains:

\[
k = \frac{\pi^2}{16\gamma} e^{2z_s/2},
\]

(70)

a relation defining implicitly \( z \) as a function of \( k \). By separating the real and imaginary parts of the logarithm of (70) one establishes that the imaginary part of \( z \) remains finite for \( k = j\Omega \), whereas its real part \( r \) tends to infinity. From the modulus of (70), one then deduces:

\[
\Omega = r \left( \frac{\pi}{16\gamma} \right) e^{\pi^2 r^2 / 2}.
\]

(71)

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